Received: *May 25, 2000*, Accepted: *July 23, 2000*

Orientifolds with discrete torsion

Matthias Klein and Raúl Rabadán

Departamento de Física Teórica C-XI and Instituto de Física Teórica C-XVI
Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain
E-Mail: matthias.klein@uam.es, rabadan@delta.ft.uam.es

Abstract: We show how discrete torsion can be implemented in $D = 4$, $\mathcal{N} = 1$ type-IIB orientifolds. Some consistency conditions are found from the closed string and open string spectrum and from tadpole cancellation. Only real values of the discrete torsion parameter are allowed, i.e. $\epsilon = \pm 1$. Orientifold models are related to real projective representations. In a similar way as complex projective representations are classified by $H^2(\Gamma, \mathbb{C}^*) = H^2(\Gamma, \mathrm{U}(1))$, real projective representations are characterized by $H^2(\Gamma, \mathbb{R}^*) = H^2(\Gamma, \mathbb{Z}_2)$. Four different types of orientifold constructions are possible. We classify these models and give the spectrum and the tadpole cancellation conditions explicitely.

Keywords: D-branes, Superstring Vacua.

Contents

JHEP07(2000)040

1. Introduction

On analyzing the effect of a non-vanishing B-field on the modular invariant partition function of an orbifold, Vafa [1] realized that there is an ambiguity related to phase factors associated to the contributions of the different twisted sectors of the partition function. It is possible to weight the contributions of the (g, h)-twisted sector, where g, h are elements of the orbifold group Γ, by a phase $\beta_{g,h}$ without spoiling modular invariance. These phases are related to elements of $H^2(\Gamma, \mathrm{U}(1))$ in such a way that if $\alpha_{g,h}$ is a 2-cocycle, the phases are of the form: $\beta_{g,h} = \alpha_{g,h}\alpha^{-1}_{h,g}$. These phases simply mean a freedom in changing the phase of some operators: the twisted fields in a sector twisted by g pick up a phase $\beta_{g,h}$ when the element h of the orbifold group is acting on them.

The effect of adding discrete torsion was analyzed in detail in some $\mathbb{Z}_N \times \mathbb{Z}_M$ heterotic orbifolds with standard embedding [2]. Some of these models have zero or negative Euler characteristic. These models are also related to some complete intersection CY manifolds.

Some examples were discused in detail by the authors of [3]. They found that the twisted fields in models with discrete torsion are related to a deformation of the orbifold singularities (in contrast to a blow-up of the singularities in the case without discrete torsion). However, some of the necessary deformations to completely smooth out the orbifold are absent, i.e. some singularities (conifolds) remain. As was commented by Vafa [1], the discrete torsion is related to some B-flux on a 2-cycle. The identification of this 2-cycle is carried out in [4]. There, a relation was found between the discrete torsion and the torsion part of the homology of the target space (more precisely, the torsion part of the blow-up of the deformed orbifold).

Discrete torsion is implemented in theories containing D-branes by using projective representations of the orbifold group [5, 6]. These projective representations are classified by $H^2(\Gamma, \mathrm{U}(1))$ in complete analogy with the discrete torsion in closed string theories found in [1]. Furthermore, the phases in projective representations appear in the amplitudes [5] as a factor $\alpha_{g,h}\alpha^{-1}_{h,g}$ where α is the cocycle. Recently [7] the relation between the closed string sector discrete torsion and the discrete torsion of the open string sector has been analyzed in disk amplitudes with a twisted closed state and a photon. In order for this amplitude to be invariant under Γ the discrete torsion β and the 2-cocycle must be related as above. D-brane charges are in correspondence with irreducible projective representations of Γ, i.e. for each irreducible representation there is a generator of the charge lattice.

The deformation that was previously analyzed in the context of closed string theories [3] can now be seen as deformations of the superpotential, changing the F-flatness conditions. As in the closed string case, some singularities remain after switching on all the possible deformations (the remaining singularities are again conifolds). The relation of models of this type with the AdS/CFT correspondence has been analyzed in [8].

The B-flux responsible for the discrete torsion parameter is related to some flux over a torsion 2-cycle. A similar problem has been analyzed in type-I strings: B-flux on the tori of an orbifold $T^4/\mathbb{Z}_N$ [9, 10, 11]. There, the new features appear at the level of untwisted tadpoles (reduction of the rank) and at the level of the closed twisted sector (some extra tensor multiplets in six dimensions). These models are T-dual to models with non-commuting Wilson Lines [12, 13, 14]. We will find some of these features also in dealing with the discrete torsion in open string theories.

Another way of understanding the discrete torsion has been realized in [15]. As an analogue of the lift of the action of the orbifold group Γ to the gauge group (orbifold Wilson lines), one can understand the discrete torsion as an ambiguity in lifting the orbifold group action to another structure (gerbes). 1-gerbes are related to 2-forms, and the action of the orbifold group on it is described in terms of the group cohomology $H^2(\Gamma, \mathrm{U}(1))$.

The aim of this article is to understand discrete torsion in orientifold models.[1] We restrict ourselves to non-compact orientifold constructions. The compact cases will be treated in a future publication [17]. In section 2, we review some of the characteristic features of discrete torsion in closed string [1] and open string theories [5]. In section 3, we analyze the D3-brane systems at an orbifold singularity in the presence of discrete torsion [5, 6]. The closed string sector, the tadpole consistency conditions and some models are studied in detail. We analyze also the deformation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ theory with discrete torsion to get the usual $\mathbb{Z}_2$ and conifold theories [18]. In section 4, non-compact orientifolds are constructed with a set of D3-branes at the orbifold singularity. In general, some D7-branes are also needed in order to cancel the Klein bottle contribution to the tadpoles. Some consistency conditions are found in different sectors: the closed string sector, the open string sector and the tadpole cancellation conditions. In all the models we consider, these three conditions lead to the same restriction on the discrete torsion parameter ϵ: only real values are allowed in the orientifold case, i.e. $\epsilon = \pm 1$. The orientifold involution is related to real projective representations. Four types of orientifold models are found. This classification is based on the possibility of having vector structure or no vector structure for each of the two generators of $\mathbb{Z}_N \times \mathbb{Z}_M$. The open string spectrum and the tadpole conditions are different for each of the four cases. We determine explicitly the open string spectrum of the D3-branes. Some of the deformations of the superpotential, that in the orbifold case are allowed, are not present in the orientifold case. In particular, only the deformation to the $\mathbb{Z}_2$ theory survives in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model.

Finally, some tools used in this paper are explained in the appendices. Many of the results of this article are based on the theory of complex and real projective representations [19, 20]. In appendix A, we therefore give a summary of their basic properties. Appendix B contains a detailed computation of the tadpole conditions for $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds. The relation between the shift formalism and the usual construction using matrices to determine the spectrum of orientifold models is explained in appendix C. We show how the shift formalism can be generalized to treat orientifolds with discrete torsion.

2. Discrete torsion

In the original paper of Vafa [1] discrete torsion appeared as phase factors in the one-loop partition function of closed string orbifold theories. If we denote by $Z(g, h)$

[1] A different approach has been taken by the authors of [16]. They discuss a compact type-I $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Their classification of possible models with unbroken supersymmetry agrees with ours.

JHEP07(2000)040

the contribution of the (g,h)-twisted sector,[2] then the total partition function Z is given by

$$Z = \frac{1}{|\Gamma|} \sum_{g,h\in\Gamma} \beta_{g,h} Z(g,h)\,, \tag{2.1}$$

where Γ is the orbifold group. The usual case without discrete torsion corresponds to $\beta_{g,h} = 1\ \forall\, g,h$. The discovery of Vafa was that non-trivial phases are consistent with modular invariance if they satisfy the following conditions:

$$\beta_{g,g} = 1\,, \qquad \beta_{g,h} = \beta_{h,g}^{-1}\,, \qquad \beta_{g,hk} = \beta_{g,h}\beta_{g,k} \quad \forall g,h,k \in \Gamma\,. \tag{2.2}$$

This implies (see appendix A) that these phases are of the form $\beta_{g,h} = \alpha_{g,h}\alpha_{h,g}^{-1}$, where α is a 2-cocycle of the group Γ. The possible discrete torsions are therefore classified by the group cohomology $H^2(\Gamma, \mathrm{U}(1))$. We will only consider abelian orbifold groups, i.e. Γ is (isomorphic to) a product of cyclic groups. Moreover, as the internal space of the string models we want to discuss is complex three-dimensional, all the possible cases can be reduced to $\Gamma = \mathbb{Z}_N$ or $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$. It is known (see e.g. [19]) that $H^2(\mathbb{Z}_N, \mathrm{U}(1)) = 1$ and $H^2(\mathbb{Z}_N \times \mathbb{Z}_M, \mathrm{U}(1)) = \mathbb{Z}_{\gcd(N,M)}$. As a consequence, discrete torsion is only possible in $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds and is generated by one element of $\mathbb{Z}_{\gcd(N,M)}$. More precisely, let g_1, g_2 be the generators of $\mathbb{Z}_N, \mathbb{Z}_M$, respectively, $p = \gcd(N,M)$ and choose $\beta_{g_1,g_2} = \omega_p^m$, with ω_p a p-th primtive root of unity and $m = 1,\ldots,p$. All the other phases $\beta_{g,h}$ are then fixed by (2.2). In the notation of appendix A, eq. (A.9), they read

$$\beta_{(a,b),(a',b')} = \epsilon^{ab'-ba'}\,, \quad \text{with } \epsilon = e^{2\pi i m/p},\ m = 1,\ldots,p\,. \tag{2.3}$$

As the partition function encodes the spectrum of the considered orbifold model, it is clear that discrete torsion modifies the spectrum. From a geometrical point of view this change in the spectrum can be understood from the fact that discrete torsion changes the cohomology of the internal space [3]. However, the spectrum of the untwisted sector remains unchanged, as can be seen from $\beta_{g,e} = 1$ (this follows from the third equation in (2.2) by setting $h = g$ and $k = e$, where e is the neutral element of Γ).

The generalization of discrete torsion to open strings has been found by Douglas [5]. The matrices γ_g that represent the action of the elements g of the orbifold group Γ on the Chan-Paton indices of the open strings form a projective representation: $\gamma_g\gamma_h = \alpha_{g,h}\gamma_{gh}$. The $\alpha_{g,h}$ are arbitrary non-zero complex numbers. They are called the factor system of the projective representation γ and they form a 2-cocycle in the sense that they satisfy (A.2). Two matrices γ_g and $\hat{\gamma}_g$ are considered projectively equivalent if there exists a non-zero complex number ρ_g such that $\hat{\gamma}_g = \rho_g\gamma_g$. As shown in appendix A the set of equivalence classes of cocycles α is $H^2(\Gamma, \mathrm{U}(1))$.

[2] By this we mean the contribution from world-sheets that are twisted by g in their space direction and by h in their time direction.

Thus, the ambiguity due to the projective representations in the open string sector is classified by the same group cohomology as the discrete torsion of [1] in the closed string sector. It is therefore natural to assume that choosing a non-trivial factor system $\alpha_{g,h}$ corresponds to discrete torsion in the open string sector. Moreover it was shown in [5, 7] that a factor system $\alpha_{g,h}$ in the open string sector of some orbifold model leads to phases $\beta_{g,h} = \alpha_{g,h}\alpha^{-1}_{h,g}$ in the closed string partition function of this model.

3. Non-compact orbifold construction

Let us consider a set of D3-branes at an orbifold singularity of the non-compact space $\mathbb{C}^3/\Gamma$, where $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$. The effect of discrete torsion in such models has been studied in [5, 6, 8]. For completeness and to fix our notations we first summarize their results and then expand on them.

3.1 Closed string spectrum

The closed string spectrum in $D = 4$ can be obtained from the cohomology of the internal space $\mathbb{C}^3/\Gamma$. Strictly speaking, this spectrum is continuous because $\mathbb{C}^3/\Gamma$ is non compact. However, as noted in [21, footnote 2], it is interesting to determine the massless spectrum that would emerge, if the internal space were compactified. This is what we want to do. The analysis is similar to the one performed by Vafa and Witten [3]. The cohomology can be split in untwisted and twisted contributions: $H^{p,q} = H^{p,q}_{\rm untw} + \sum_{g\in\Gamma\setminus\{e\}} H^{p,q}_g$. The untwisted Hodge number $h^{p,q}_{\rm untw} = \dim(H^{p,q}_{\rm untw})$ is just given by the number of Γ-invariant (p,q)-forms on $\mathbb{C}^3$. This result is independent of the discrete torsion. The twisted contributions are due to the singularities of $\mathbb{C}^3/\Gamma$. In the sector twisted by g, one has to find the Γ-invariant forms that can be defined on the subspace $\mathcal{M}_g$ of $\mathbb{C}^3$ that is fixed under the action of g. If discrete torsion is present, then the forms on $\mathcal{M}_g$ must be invariant under the action of h combined with a multiplication by $\beta_{g,h}$ $\forall\, h \in \Gamma$. In the non-compact case that we are treating here, $\mathcal{M}_g$ is either a point at the origin of $\mathbb{C}^3$ or a complex plane located at the origin of the transverse $\mathbb{C}^2$. In the former case there is only a $(0,0)$-form. If no discrete torsion is present, it is invariant and contributes one unit to $h^{1,1}_g$ or to $h^{2,2}_g$ depending on the specific model. In the case with discrete torsion, there is no contribution to the cohomology from this sector. If $\mathcal{M}_g$ is a complex plane, then, in the case without discrete torsion, the $(0,0)$-form and the $(1,1)$-form are invariant. They contribute one unit to $h^{1,1}_g$ and to $h^{2,2}_g$. In the case with discrete torsion the $(0,0)$-form and the $(1,1)$-form are invariant if $\beta_{g,h} = 1$ $\forall\, h$. The $(0,1)$-form and the $(1,0)$-form are invariant if the action of h multiplies these forms by a phase which is opposite to $\beta_{g,h}$ $\forall\, h$. In this case, they contribute one unit to $h^{1,2}_g$ and to $h^{2,1}_g$.

For simplicity let us first take $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ [5] and then indicate the generalization to $\mathbb{Z}_N \times \mathbb{Z}_M$. Using the method explained above, the Hodge diamond of the untwisted cohomology is found to be

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

For all the other $\mathbb{Z}_N \times \mathbb{Z}_M$ models, one finds nearly the same untwisted cohomology, the only difference being that in those cases $h^{2,1}_{\text{untw}} = h^{1,2}_{\text{untw}} = 0$.

There are three twisted sectors in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to the three non-trivial elements (0,1), (1,0), (1,1), where we used the notation introduced in appendix A (below eq. (A.8)). Each of these sectors gives the same contribution to the twisted cohomology. In the case without discrete torsion:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

In the case with discrete torsion:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
0 & & 1 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

For a general $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold one finds that, only if minimal discrete torsion $\epsilon = \omega_{\gcd(N,M)}$ (i.e. $m = 1$ in (2.3)) is present, is there a non-zero contribution to $h^{1,2}_{\text{tw}} = \sum_{g \in \Gamma \backslash \{e\}} h^{1,2}_g$. Three cases can be distinguished:

(i) $N = M \Rightarrow h^{1,2}_{\text{tw}} = h^{2,1}_{\text{tw}} = 3$

(ii) $N = \gcd(N, M) < M \Rightarrow h^{1,2}_{\text{tw}} = h^{2,1}_{\text{tw}} = 2$

(iii) $N \neq \gcd(N, M) \neq M \Rightarrow h^{1,2}_{\text{tw}} = h^{2,1}_{\text{tw}} = 0$.

In order to obtain the closed string spectrum in $D = 4$, one has to dimensionally reduce the massless spectrum of type-IIB supergravity in $D = 10$: the metric $g^{(10)}$, the NSNS 2-form $B^{(10)}$, the dilaton $\phi^{(10)}$, the RR-forms $C_0^{(10)}$, $C_2^{(10)}$, $C_4^{(10)}$. (We only give the bosons, the fermions are related to them by supersymmetry.) This is done by contracting their Lorentz indices with the differential forms of the internal space. The resulting spectrum has $\mathcal{N} = 2$ supersymmetry in $D = 4$. For a general configuration we get:

- $g^{(4)}$, $(h^{1,1} + 2h^{2,1})$ scalars (from $g^{(10)}$)
- a 2-form, $h^{1,1}$ scalars (from $B^{(10)}$)
- a scalar (from $\phi^{(10)}$)
- a scalar (from the $C_0^{(10)}$)
- a 2-form, $h^{1,1}$ scalars (from $C_2^{(10)}$)
- $(h^{2,1} + 1)$ vectors, $h^{1,1}$ 2-forms (from $C_4^{(10)}$).

These fit into the following $\mathcal{N} = 2$ SUSY representations (see e.g. [22]):

- a gravity multiplet (consisting of $g^{(4)}$, a vector and fermions)
- a double tensor multiplet (consisting of two 2-forms, two scalars and fermions)
- $h^{1,1}$ tensor multiplets (consisting of a 2-form, three scalars and fermions)
- $h^{2,1}$ vector multiplets (consisting of a vector, two scalars and fermions)

For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example we find (in $\mathcal{N} = 2$ multiplets):

- without discrete torsion: gravity, a double tensor, 3 vectors and 6 tensors
- with discrete torsion: gravity, a double tensor, 6 vectors and 3 tensors

3.2 Tadpoles

In the orbifold case, only oriented surfaces appear in computing the tadpoles. There is only one involving branes: the cylinder. As the space is non-compact all the tadpoles with an inverse dependence on the volume of the internal coordinates vanish. These are the untwisted ones and those corresponding to twisted sectors that leave one complex plane fixed. So there is no restriction on the total number of branes and on $\mathrm{Tr}\,\gamma_g$ if g has a fixed plane. This means that, for example, in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case there is no restriction from tadpoles (that is the case of ref. [5]).

In general, one can add D7_i-branes to the system of D3-branes at the singularity, where the index $i = 1, 2, 3$ refers to the complex plane with Dirichlet conditions. Then the cylinder amplitude can be split into four sectors: 33, $7_i 7_i$, $7_i 7_j$ and $7_i 3$.

It is convenient to denote a group element $g = g_1^n g_2^m$ of $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$ by the 2-vector $\bar{k} = (n, m)$. Similarly, we form 2-vectors of the components of the twist vectors v_i and w_i that represent the action of g_1 and g_2 on the i-th complex plane: $\bar{v}_i = (v_i, w_i)$. If we define $s_i = \sin(\pi \bar{k} \cdot \bar{v}_i)$, then the cylinder contribution is of the form (see appendix B):

$$\mathcal{C} = \sum_{\bar{k}=(1,1)}^{(N,M)} \frac{1}{8 s_1 s_2 s_3} \left[8 s_1 s_2 s_3 \operatorname{Tr} \gamma_{\bar{k},3} + \sum_{i=1}^{3} 2 s_i \operatorname{Tr} \gamma_{\bar{k},7_i} \right]^2 . \tag{3.1}$$

As D7-branes are not needed for the consistency of the orbifold models, we restrict ourselves to configurations without D7-branes. In this case the tadpole conditions are summarized by:

$$\operatorname{Tr} \gamma_g = 0 \,, \quad \text{if } g \text{ has no fixed planes} \,. \tag{3.2}$$

As we have seen, discrete torsion appears in two ways: as phase factors in the closed string partition function and as a non-trivial factor system in the projective representation of the orbifold group on the Chan-Paton indices of open strings. The former could only modify the Klein bottle diagram the latter could only affect the Möbius strip. Both diagrams are not present in the orbifold case. Consequently, in this case, the tadpole conditions are not changed by discrete torsion.

Using formula (A.14) for the characters of a projective representation, it is easy to find solutions to the tadpole conditions (3.2). According to the results of appendix A, the matrix γ_g (with $g = g_1^a g_2^b$) of a general projective representation with discrete torsion ϵ is of the form

$$\bigoplus_{k,l} \left(\sqrt{\epsilon}\right)^{-ab} \left(\omega_N^k \gamma_{g_1}\right)^a \left(\omega_M^l \gamma_{g_2}\right)^b \otimes \mathbb{1}_{n_{kl}} \,, \tag{3.3}$$

where γ_{g_1} and γ_{g_2} are given in (A.11), $k = 0, \ldots, N/s - 1$, $l = 0, \ldots, M/s - 1$. We recall that $\gamma_{g_{1/2}}$ are $(s \times s)$-matrices, where s is the smallest positive integer, such that $\epsilon^s = 1$. One can readily verify that the regular representation, i.e. $n_{kl} = s \; \forall \, k, l$, is a solution of (3.2). But there are many more solutions. For example each set of n_{kl} that either only depends on k or only depends on l (i.e. $n_{kl} = \tilde{n}_k \; \forall \, l$ or $n_{kl} = \tilde{n}_l \; \forall \, k$) is possible.

To make some further statements, we need to specify how the generators g_1, g_2 of $\mathbb{Z}_N \times \mathbb{Z}_M$ act on the three complex coordinates of the internal space. We choose the following action:

$$\begin{aligned} g_1 &: (z_1, z_2, z_3) \longrightarrow \left(e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3\right) , \\ g_2 &: (z_1, z_2, z_3) \longrightarrow \left(e^{2\pi i w_1} z_1, e^{2\pi i w_2} z_2, e^{2\pi i w_3} z_3\right) \\ &\text{with } v = \frac{1}{N}(1, -1, 0) \,, \qquad w = \frac{1}{M}(0, 1, -1) \,. \end{aligned} \tag{3.4}$$

For $N = M$, all possible actions of g_1, g_2 can be brought to the form (3.4) by permuting the elements of $\mathbb{Z}_N \times \mathbb{Z}_N$. If $N \neq M$, one can choose different (non-equivalent) actions for g_1, g_2. We restrict ourselves to one alternative:

$$
\begin{aligned}
&g_1 : (z_1, z_2, z_3) \longrightarrow \left(e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3\right), \\
&g_2 : (z_1, z_2, z_3) \longrightarrow \left(e^{2\pi i w'_1} z_1, e^{2\pi i w'_2} z_2, e^{2\pi i w'_3} z_3\right) \\
&\text{with } v = \frac{1}{N}(1, -1, 0), \qquad w' = \frac{1}{M}(1, -2, 1).
\end{aligned} \tag{3.5}
$$

To distinguish the two different actions, we will denote the orbifold corresponding to the latter by $\mathbb{Z}_N \times \mathbb{Z}'_M$.

Two interesting cases are (i) $N = M$ and (ii) M a multiple of N. In the first case, there is no tadpole condition for $s = N$ and $s = N/2$ (the only non-vanishing characters correspond to group elements that have fixed planes, see eq. (A.14)). So the first condition arises for $s = N/3$, which only for $N \geq 6$ leads to an orbifold with discrete torsion. The condition that the n_{kl} of (3.3) have to satisfy for $N = M = 6$ and $s = 2$ is $\sum_{k,l=0}^{2} n_{kl} e^{2\pi i(k+2l)/3} = 0$. As shown above, it is easy to find solutions for the n_{kl}. For higher N more conditions of this type have to be satisfied. If M is a multiple of N, then there are non-vanishing characters even in the case of minimal discrete torsion, $s = N$: $\mathrm{Tr}\, \gamma_{(0,cN)} = N \sum_{l=0}^{M/N-1} n_{0,l}\, \omega_M^{lcN}$, where $c = 1, \ldots, M/N - 1$. However, the group element $(0, cN)$ has a fixed plane if the group action is as in (3.4). Thus, only for the $\mathbb{Z}_N \times \mathbb{Z}'_M$ orbifold do we get a tadpole condition in the case $s = N$. A solution to this condition is to take $n_{0,l} = n \;\forall\, l$. If M/N is a prime number, then this solution is unique, else different choices for $n_{0,l}$ are possible.

3.3 Open string spectrum

Discrete torsion is implemented in the relation between the elements γ_g of the representation of the group acting on the Chan-Paton matrices [5]:

$$
\gamma_g \gamma_h = \alpha_{g,h} \gamma_{gh} \,. \tag{3.6}
$$

As in the case without discrete torsion, one gets the gauge fields $\lambda^{(0)}$ and matter fields $\lambda^{(i)}$, $i = 1, 2, 3$, taking the solutions to the projections:

$$
\gamma_g^{-1} \lambda \gamma_g = r(g) \lambda \,, \tag{3.7}
$$

where $r(g)$ is the matrix that represents the action of g on $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$.

Let us first treat the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model of [5] in detail and then show how this is generalized to more complicated models. The only non-trivial phases in (3.6) are

$$
\alpha_{g,h} = -\alpha_{h,g} = i \,, \tag{3.8}
$$

where we have denoted the generators of the first and second $\mathbb{Z}_2$ by g and h. This corresponds to a projective representation with discrete torsion $\epsilon = \alpha_{g,h}\alpha_{h,g}^{-1} = -1$. From (3.3), we find that the matrices are of the form:

$$\gamma_e = \mathbb{1}_2 \otimes \mathbb{1}_n\,, \qquad \gamma_g = \sigma_3 \otimes \mathbb{1}_n\,, \qquad \gamma_h = \sigma_1 \otimes \mathbb{1}_n\,, \qquad \gamma_{gh} = \sigma_2 \otimes \mathbb{1}_n\,, \tag{3.9}$$

where σ_i are the Pauli matrices and n is an arbitrary parameter (it counts the number of dynamical D-branes).

The solution to (3.7) is given by

$$\lambda^{(0)} = \mathbb{1}_2 \otimes X\,, \qquad \lambda^{(i)} = \sigma_i \otimes Z_i\,, \tag{3.10}$$

where X, Z_i are arbitrary $(n \times n)$-matrices. This corresponds to gauge group $\mathrm{U}(n)$ and three adjoint matter fields Z_1, Z_2, Z_3 in $\mathcal{N} = 1$ multiplets. The superpotential can be obtained in the usual way and reads:

$$W = \mathrm{Tr}(Z_1 Z_2 Z_3 + Z_2 Z_1 Z_3)\,. \tag{3.11}$$

It has to be completed by the deformations

$$\Delta W = \sum_{i=1}^{3} \zeta_i \,\mathrm{Tr}\, Z_i\,, \tag{3.12}$$

where ζ_i are the twisted modes from the closed string spectrum.

For general γ-matrices (3.3), representing the action of $\mathbb{Z}_N \times \mathbb{Z}_M$ on the open strings, the solution to (3.7) is a gauge group

$$\prod_{k=0}^{N/s-1} \prod_{l=0}^{M/s-1} \mathrm{U}(n_{kl}) \tag{3.13}$$

with matter in adjoint and bifundamental representations. For some models the tadpole conditions impose restrictions on the numbers n_{kl}. The spectrum can easiest be obtained from the corresponding quiver diagram, as explained in [8]. A more rigorous way to find the spectrum uses the shift formalism developed in [23]. We explain this method in appendix C. Note that a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold with discrete torsion characterized by the integer[3] s has the same open string spectrum as the orbifold $\mathbb{Z}_{N/s} \times \mathbb{Z}_{M/s}$ without discrete torsion. The only difference appears in the superpotential where some of the terms acquire additional phases [8].

In table 1 we present the explicit solution for some models.

[3] Recall that s was defined as the smallest positive integer, such that $\epsilon^s = 1$.

JHEP07(2000)040

Γ	torsion	gauge group	matter
$\mathbb{Z}_2 \times \mathbb{Z}_2$	-1	$\mathrm{U}(n)$	3 adj
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$e^{2\pi i/3}$	$\mathrm{U}(n)$	3 adj
$\mathbb{Z}_2 \times \mathbb{Z}_4$	-1	$\mathrm{U}(n_1) \times \mathrm{U}(n_2)$	$(\mathrm{adj}, \mathbb{1}) + (\mathbb{1}, \mathrm{adj}) +$ $+2(\Box, \overline{\Box}) + 2(\overline{\Box}, \Box)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	i	$\mathrm{U}(n)$	3 adj
$\mathbb{Z}_4 \times \mathbb{Z}_4$	-1	$\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times$ $\times \mathrm{U}(n_3) \times \mathrm{U}(n_4)$	$(\Box, \overline{\Box}, \mathbb{1}, \mathbb{1}) + (\mathbb{1}, \Box, \overline{\Box}, \mathbb{1}) + (\mathbb{1}, \mathbb{1}, \Box, \overline{\Box}) +$ $+(\Box, \mathbb{1}, \overline{\Box}, \mathbb{1}) + (\mathbb{1}, \Box, \mathbb{1}, \overline{\Box}) + (\Box, \mathbb{1}, \mathbb{1}, \overline{\Box}) +$ $+$ conjugate
$\mathbb{Z}_2 \times \mathbb{Z}_6$	-1	$\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3)$	$(\Box, \overline{\Box}, \mathbb{1}) + (\Box, \mathbb{1}, \overline{\Box}) + (\mathbb{1}, \Box, \overline{\Box}) +$ $+(\overline{\Box}, \Box, \mathbb{1}) + (\overline{\Box}, \mathbb{1}, \Box) + (\mathbb{1}, \overline{\Box}, \Box) +$ $+(\mathrm{adj}, \mathbb{1}, \mathbb{1}) + (\mathbb{1}, \mathrm{adj}, \mathbb{1}) + (\mathbb{1}, \mathbb{1}, \mathrm{adj})$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	-1	$\mathrm{U}(n) \times \mathrm{U}(n) \times \mathrm{U}(n)$	$3(\overline{\Box}, \Box, \mathbb{1}) + 3(\Box, \mathbb{1}, \overline{\Box}) + 3(\mathbb{1}, \overline{\Box}, \Box)$
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$e^{2\pi i/3}$	$\mathrm{U}(n_1) \times \mathrm{U}(n_2)$	$(\mathrm{adj}, \mathbb{1}) + (\mathbb{1}, \mathrm{adj}) +$ $+\, 2(\Box, \overline{\Box}) + 2(\overline{\Box}, \Box)$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$e^{2\pi i/6}$	$\mathrm{U}(n)$	3 adj
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$e^{2\pi i/3}$	$\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times$ $\times \mathrm{U}(n_3) \times \mathrm{U}(n_4)$	$(\Box, \overline{\Box}, \mathbb{1}, \mathbb{1}) + (\mathbb{1}, \Box, \overline{\Box}, \mathbb{1}) + (\mathbb{1}, \mathbb{1}, \Box, \overline{\Box}) +$ $+(\Box, \mathbb{1}, \overline{\Box}, \mathbb{1}) + (\mathbb{1}, \Box, \mathbb{1}, \overline{\Box}) + (\Box, \mathbb{1}, \mathbb{1}, \overline{\Box}) +$ $+$ conjugate
$\mathbb{Z}_6 \times \mathbb{Z}_6$	-1	$\mathrm{U}(n_1)^3 \times \mathrm{U}(n_2)^3 \times \mathrm{U}(n_3)^3$	$(\underline{\underline{\Box, \mathbb{1}, \mathbb{1}}; \mathbb{1}, \mathbb{1}, \mathbb{1}; \underline{\overline{\Box}, \mathbb{1}, \mathbb{1}}}) +$ $+(\underline{\underline{\Box, \overline{\Box}, \mathbb{1}}; \mathbb{1}, \mathbb{1}, \mathbb{1}; \mathbb{1}, \mathbb{1}, \mathbb{1}}) +$ $+(\underline{\underline{\Box, \mathbb{1}, \mathbb{1}}; \underline{\mathbb{1}, \mathbb{1}, \overline{\Box}}; \mathbb{1}, \mathbb{1}, \mathbb{1}})$

Table 1: Open string spectrum of some non-compact orbifolds with discrete torsion. The conjugate representations are obtained exchanging fundamentals and antifundamentals. Underlining stands for all cyclic permutations of the underlined elements (first line) respectively cyclic permutations of groups of three entries (second line). If two groups of entries are underlined, they are permuted simultaneously and not independently. Thus, in the case of $\mathbb{Z}_6 \times \mathbb{Z}_6$, $\epsilon = -1$, one has 27 matter representations.

Note that the $\mathbb{Z}_2 \times \mathbb{Z}_6'$ example with discrete torsion $\epsilon = -1$ would have non-abelian gauge anomalies if the gauge group were $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3)$, with $n_1 \neq n_2 \neq n_3$. However, as mentioned in the previous subsection, the requirement of tadpole cancellation implies that $n_1 = n_2 = n_3$.[4] This can be seen as follows. The action of $\mathbb{Z}_2 \times \mathbb{Z}_6'$ on the complex coordinates is as in (3.5). There are three twisted sectors (we are not taking into account the inverse because they give the same tadpole cancellation condition) that do not have any fixed plane: h, h^2 and gh, where g and h are the generators of $\mathbb{Z}_2$ and $\mathbb{Z}_6'$. Three tadpoles must be cancelled:

$$\mathrm{Tr}\, \gamma_h = 0\,, \qquad \mathrm{Tr}\, \gamma_{gh} = 0\,, \qquad \mathrm{Tr}\, \gamma_{h^2} = 0\,. \tag{3.14}$$

The action of the orbifold group on the Chan-Paton indices can be chosen as

$$\gamma_h = \mathrm{diag}\left(\mathbb{1}_{n_1}, -\mathbb{1}_{n_1}, e^{2\pi i/6}\mathbb{1}_{n_2}, e^{8\pi i/6}\mathbb{1}_{n_2}, e^{4\pi i/6}\mathbb{1}_{n_3}, e^{10\pi i/6}\mathbb{1}_{n_3}\right),$$

[4]The relation between anomaly freedom and tadpole cancellation has been discussed in [24, 25].

$$\gamma_g = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} . \tag{3.15}$$

The first two tadpole conditions are inmediatly satisfied but the third one implies:

$$n_1 + e^{2\pi i/3} n_2 + e^{-2\pi i/3} n_3 = 0 \, . \tag{3.16}$$

The solution coincides with the condition for non-abelian gauge anomaly cancellation: $n_1 = n_2 = n_3$.

A similar calculation shows that, in the case of $\mathbb{Z}_6 \times \mathbb{Z}_6$ with discrete torsion $\epsilon = -1$, the gauge group is not of the most general form (3.13). One needs $n_{1,1} = n_{1,2} = n_{1,3}$, $n_{2,1} = n_{2,2} = n_{2,3}$ and $n_{3,1} = n_{3,2} = n_{3,3}$ for tadpole cancellation. The same condition can also be deduced from the requirement of anomaly freedom.

3.4 Resolution of the singularities

D-branes are interesting probes of geometry and topology. World volume theories of D-branes at a singularity are related to the structure of this singularity. Spacetime can even be understood as a derived concept, emerging from non-trivial moduli spaces of the theory on the D-brane [21, 26].

In our case, there is a correspondence between the moduli space of the theory and the transverse space to the D3-branes. This correspondence allows a map between different theories and different singularities (this has been extensively studied, see e.g. [27, 18]).

Cases without discrete torsion are related to deformations of the D-flatness conditions. Fayet-Iliopoulos terms are controlled by the twisted sector moduli. Non-trivial values of the Fayet-Iliopoulos parameters can be interpreted as resolutions of the singularities transverse to the branes. Arbitrary values of these terms resolve completely the singularity.

Cases with discrete torsion are related to deformations of the F-flatness conditions [5, 6]. The singularities are not smoothed out by blow-ups but by deformations [3]. However, these singularities cannot be completely smoothed out due to the lack of a sufficient number of twisted states. Some singularities are remaining.

Resolutions and deformations are the two main strategies of desingularization of orbifolds. An analysis of the different desingularizations associated to some orbifold singularities is done in [28]. Only a small number of the possible desingularizations are available in string theory.

In this section we want to find the field theories related to the deformation of the $\mathbb{C}/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity with discrete torsion $\epsilon = -1$. The analysis in the orientifold case will be derived from this one.

From the three adjoint fields one can construct the $\mathrm{SU}(n)$ invariants. We consider the diagonal $\mathrm{U}(1)$ as decoupled [29]. These $\mathrm{SU}(n)$ invariants are of the form:

$$M_{i,j} = \mathrm{Tr}(Z_i Z_j)\,, \qquad B = \mathrm{Tr}(Z_1[Z_2, Z_3])\,. \tag{3.17}$$

They satisfy the following relation:

$$B^2 = \det M\,. \tag{3.18}$$

Using the F-flatness conditions coming from the superpotential (3.11) and (3.12) above, one can obtain the following relation between the invariants:

$$\det\begin{pmatrix} M_{11} & \zeta_3 & \zeta_2 \\ \zeta_3 & M_{22} & \zeta_1 \\ \zeta_2 & \zeta_1 & M_{33} \end{pmatrix} = B^2\,. \tag{3.19}$$

There are three different singularities derived from this case:

(i) If all the ζ_i are equal to zero, the above equation between the invariants is of the form:

$$M_{11}M_{22}M_{33} = B^2\,. \tag{3.20}$$

This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity.

(ii) If only one of the three ζ_i, say ζ_1, is different from zero, the equation relating the invariants is:

$$M_{11}\left(M_{22}M_{33} - \zeta_1^2\right) = B^2\,. \tag{3.21}$$

The singularity is at the points $(M_{22}M_{33} - \zeta_1^2, M_{11}M_{33}, M_{11}M_{22}, -2B) = (0,0,0,0)$. This corresponds to a $\mathbb{C}^*$ of singularities. Changing the variables, such that the singularity will pass through the origin, one has:

$$M_{22} = \zeta_1 + x + y\,, \qquad M_{33} = \zeta_1 + x - y\,. \tag{3.22}$$

Taking the lowest order in the polynomial (just for analyzing the points near the singularity), gives:

$$M_{11}(2x\zeta_1^2) = B^2\,. \tag{3.23}$$

That is a $\mathbb{Z}_2$ singularity. In order to obtain the field theory, let us give the following vev's to the fields:

$$\langle Z_1\rangle = 0\,, \qquad \langle Z_2\rangle = v\sigma_3\,, \qquad \langle Z_3\rangle = v\sigma_3\,. \tag{3.24}$$

Plugging these values back into the F-flatness conditions, allows us to get the deformations:

$$\zeta_2 = \zeta_3 = 0\,, \qquad \zeta_1 = 2v^2\,. \tag{3.25}$$

These vev's break the group down to $\mathrm{U}(n/2)\times\mathrm{U}(n/2)$. The remaining spectrum is:

	U$(n/2)$	U$(n/2)$
ϕ_1	adj	1
ϕ_2	1	adj
A_{12}	$\square$	$\overline{\square}$
B_{12}	$\square$	$\overline{\square}$
A_{21}	$\overline{\square}$	$\square$
B_{21}	$\overline{\square}$	$\square$

The A_{12}, B_{12}, A_{21}, B_{21}, ϕ_1 and ϕ_2 can be seen as the components of the Z_i matrices:

$$Z_1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{3.26}$$

$$Z_2 = v\sigma_3 + \begin{pmatrix} (M_1+\phi_1)/2 & 2A_{12} \\ 2A_{21} & (M_2+\phi_2)/2 \end{pmatrix}, \tag{3.27}$$

$$Z_3 = v\sigma_3 + \begin{pmatrix} (M_1-\phi_1)/2 & -2A_{12} \\ -2A_{21} & (M_2-\phi_2)/2 \end{pmatrix}. \tag{3.28}$$

The fields M_i and B_{ii} get masses. The relation between the non-diagonal components of the Z_2 and Z_3 fields comes from the degrees of freedom that are eaten by the Higgs mechanism. The superpotential for the massless fields is obtained by integrating out the massive states:

$$W = -2\,\mathrm{Tr}\left[(\phi_1+\phi_2)(A_{12}B_{21}+A_{21}B_{12})\right]. \tag{3.29}$$

(iii) Let us now take all the three ζ_i different from zero. For that, one can analyze how are the possible deformations of the $\mathbb{Z}_2$ case above (another possibility with the same result is taking the vev's of the fields all proportional to σ_3). Let us take for simplicity U(1) $\times$ U(1) as the gauge group at the $\mathbb{Z}_2$ singularity. The deformations can be expressed in terms of the massless fields using the F-flatness conditions. There are four possible solutions:

$$\Delta W = \pm 2v^2\left[4v + \frac{8}{v}A_{12}A_{21} + \frac{1}{4v}\left(\phi_1^2+\phi_2^2\right)\right], \tag{3.30}$$

$$\Delta W = \pm 2v^2\frac{1}{4v}\left(\phi_1^2-\phi_2^2\right). \tag{3.31}$$

The second one is the deformation considered by Klebanov and Witten [18] that takes to the theory at the conifold. The superpotential gives masses to the two adjoints, leaving as massless spectrum:

	U$(n/2)$	U$(n/2)$
A_{12}	$\square$	$\overline{\square}$
B_{12}	$\square$	$\overline{\square}$
A_{21}	$\overline{\square}$	$\square$
B_{21}	$\overline{\square}$	$\square$

4. Non-compact orientifold construction

Let us locate a set of D3-branes at an orbifold singularity of $\mathbb{C}^3/\Gamma$. The action of the orbifold group Γ permutes the branes in an appropriate way [21]. We take an orientifold of the form $\Omega' = \Omega(-1)^{F_L} R_1 R_2 R_3$ in order to preserve $\mathcal{N}=1$ supersymmetry in the effective theory in four dimensions (A models in [30]). The action of Ω on the Chan-Paton matrices can be chosen to be either symmetric (A1 models) or antisymmetric (A2 models). In general these models will need the presence of D7-branes for consistency. This can be deduced from the tadpole cancellation conditions.

4.1 Closed string spectrum

As in the orbifold case, the closed string spectrum is continuous because the space $\mathbb{C}^3/\Gamma$ is non compact. Again, the philosophy is to find the spectrum that would exist if the internal space were compactified. The Ω parity is accompanied by a J operation that relates states from one twisted sector with states from the inverse sector. In order to obtain the ΩJ invariant states one must combine the states from the g and g^{-1} twisted sectors. Because of this, the counting of states is different compared to the orbifold case.

For sectors that are invariant under the J permutation, i.e. the untwisted sector and the order-two twisted sectors, the spectrum in four dimensions is obtained by dimensionally reducing the fields of the type-I string in ten dimensions. These fields are the metric $g^{(10)}$ and the dilaton $\phi^{(10)}$ in the NSNS sector and the 2-form $C_2^{(10)}$ in the RR sector. Their Lorentz indices have to be contracted with the differential forms of the untwisted and order-two twisted cohomology (see section 3.1). In $\mathcal{N}=1$ representations these sectors give: the gravity multiplet, a linear multiplet and $(h^{1,1}+h^{2,1})_{\text{untw+order-two}}$ chiral multiplets.

One can split the remaining sectors into two types:

- The sectors twisted by g that give the same contribution to the cohomology as their inverse sectors, i.e. twisted by g^{-1} (these sectors always have fixed planes). Together they give $h_g^{1,1}$ linear multiplets, $h_g^{2,1}$ vectors and $h_g^{1,1}+h_g^{2,1}$ chiral multiplets [31].

- The sectors twisted by g that give a different contribution to the cohomology as their inverse sectors. Only one combination of the fields survives: $h_g^{1,1}$ linear multiplets and $h_g^{2,1}$ vectors. If the contribution from the g-twisted sector to $h_g^{1,2}$ and $h_g^{2,1}$ is not the same (that happens if the discrete torsion is different from ± 1), one cannot match properly the states from one sector to the inverse sector. An orientifold model that contains such a sector is ill-defined.

Γ	torsion	closed string spectrum
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	$1\ G + 1\ L + 9\ \Phi$
	-1	$1\ G + 1\ L + 9\ \Phi$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	$1\ G + 5\ L + 6\ \Phi$
	$e^{\pm 2\pi i/3}$	not well defined

Table 2: Closed string spectrum for the orientifolds $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ with and without discrete torsion . The spectrum is organized in $\mathcal{N} = 1$ representations (G is the gravitational multiplet, L a linear multiplet and Φ a chiral multiplet).

> This consistency criterium agrees with the one we will find in the open string sector. There, it turns out that only real values of the discrete torsion are allowed.[5]

In table 2, one can compare the $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ closed string spectrum. The first one admits the values $\epsilon = \pm 1$ for the discrete torsion. Note that, although the cohomology is different in both cases, the closed string spectrum is the same for both values of the discrete torsion. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ model has $H^2(\Gamma, \mathrm{U}(1)) = \mathbb{Z}_3$, that is $\epsilon = e^{\pm 2\pi i/3}$ or $\epsilon = 1$. Only the last value is allowed in the orientifold case.

Let us explain in detail why the closed string spectrum cannot be consistently defined if $h_g^{2,1} \neq h_g^{1,2}$ for a sector twisted by g. The states corresponding to this sector can be obtained from the shift formalism.[6] Take one sector g with $h_g^{1,2} \neq 0$ and $h_g^{2,1} = 0$. This gives $h_g^{1,2}$ helicity components $+1$ of some vectors and none with helicity -1. In the inverse sector g^{-1} one gets $h_{g^{-1}}^{1,2} = 0$ and $h_{g^{-1}}^{2,1} = h_g^{1,2}$. This leads to $h_{g^{-1}}^{2,1}$ helicity components -1 of some vectors. In type-IIB theory this causes no problem, we get $h_g^{1,2}$ vectors. The problem arises if we want to impose the J projection that relates the g-twisted sector with its inverse: one cannot match the states from these two sectors because they have opposite helicity.

This rules out several orientifold models: $\mathbb{Z}_3 \times \mathbb{Z}_3$ with discrete torsion, $\mathbb{Z}_4 \times \mathbb{Z}_4$ with discrete torsion $\pm i$, $\mathbb{Z}_3 \times \mathbb{Z}_6$ with discrete torsion, etc. One can check that the orientifold projection is compatible with discrete torsion only for the values $\epsilon = +1$ and $\epsilon = -1$. This can be seen from the formulae for $h_g^{1,2}$ and $h_g^{2,1}$ in the compact cases [17]:

$$h_g^{1,2} = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{g,h}\, \widetilde{\chi}(g,h) e^{2\pi i v_h(g)}\,, \qquad h_g^{2,1} = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{g,h}\, \widetilde{\chi}(g,h) e^{-2\pi i v_h(g)}\,. \tag{4.1}$$

Only if the discrete torsion is real, are $h_g^{1,2}$ and $h_g^{2,1}$ equal for any twist. We will see that this criterium in the closed string sector coincides with the one coming from the open string sector related to projective representations.

[5]We thank Angel Uranga for pointing out to us that this can also be seen by noticing that ΩJ is not a symmetry of type-IIB theory in the case of non-real discrete torsion.

[6]This will be explained in detail in [17].

4.2 Tadpoles

As stated above, discrete torsion appears as a phase between different twisted sectors in closed string theories and as a phase between the matrices that represent the action of Γ on the Chan-Paton indices in the open string sector. In order to compute the tadpole contribution, one must sum over three different diagrams: the cylinder ($\mathcal{C}$), the Möbius strip ($\mathcal{M}$) and the Klein bottle ($\mathcal{K}$). The first two diagrams contain a trace over the projective representation γ. Discrete torsion phases appear in the Möbius strip and in the Klein bottle. The three diagrams can be written as a sum over twisted sectors:

$$\mathcal{C} = \sum_{g\in\Gamma} \tilde{\mathcal{C}}(g)\,, \qquad \mathcal{M} = \sum_{g\in\Gamma} \tilde{\mathcal{M}}(g)\,, \qquad \mathcal{K} = \sum_{g,h\in\Gamma} \tilde{\epsilon}_h\, \beta_{h,g}\, \tilde{\mathcal{K}}_h(g)\,,$$
$$\text{with} \quad \tilde{\mathcal{C}}(g) \propto (\mathrm{Tr}\,\gamma_g)^2 \quad \text{and} \quad \tilde{\mathcal{M}}(g) \propto \mathrm{Tr}\left(\gamma_{\Omega g}^{-1}\gamma_{\Omega g}^{\top}\right). \tag{4.2}$$

The phases $\beta_{h,g}$ satisfy (2.2) and are related to the discrete torsion. The additional factor $\tilde{\epsilon}_h = \pm 1$ appears because of the possibility of having vector structure or not in each of the two factors of $\mathbb{Z}_N \times \mathbb{Z}_M$. As J relates the orbifold elements h and h^{-1}, the only non-vanishing contributions to the Klein bottle are (see appendix of [23]) $\tilde{\mathcal{K}}_e(g)$ and $\tilde{\mathcal{K}}_{h^{(2)}}(g)$, where e is the neutral element of Γ and $h^{(2)}$ is of order two, i.e. $(h^{(2)})^2 = e$. The first of these two contributions is not modified by discrete torsion, because (2.2) implies $\beta_{e,g} = 1$. The second contribution depends on discrete torsion, as explained in appendix B.

In general, models of this type require the presence of D7-branes. The tadpole calculation is detailed in appendix B. To give the result of this calculation, we first introduce some notation. It is convenient to denote the group elements of $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$ by $\bar{k} = (n, m)$ as in eq. (3.1). One has to distinguish even twists $\bar{k}$, i.e. there is a $\bar{k}' \in \Gamma$, such that $\bar{k} = 2\bar{k}'$, from odd twists that cannot be written in this form. If N is odd and M is even, there exists one order-two element $\bar{k}_1$. If N and M are both even there are three order-two elements $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, where the index denotes the complex plane that is fixed by this element (i.e. $\bar{k}_1 = (0, M/2)$, etc). As will be explained below, the generalization of the notion of vector structure to $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds leads to four different boundary conditions on the γ matrices: $\gamma_{g_1}^N = \pm\mathbb{1}$, $\gamma_{g_2}^M = \pm\mathbb{1}$. We define the numbers $\mu_i = \pm 1$ (the index i refers to the complex plane that is fixed by the corresponding group element) by

$$\gamma_{g_1}^N = \mu_3\, \mathbb{1}\,, \qquad \gamma_{g_2}^M = \mu_1\, \mathbb{1}\,, \qquad (\gamma_{g_1}\gamma_{g_2})^{\mathrm{lcm}(N,M)} = \mu_2 \mathbb{1}\,. \tag{4.3}$$

Of course, only two of the μ_i are independent. One can show that

$$\mu_2 = \begin{cases} \epsilon^{-MN/4}\, \mu_3\, \mu_1 & \text{if } \frac{M}{\gcd(N,M)} \text{ and } \frac{N}{\gcd(N,M)} \text{ odd} \\ \mu_1 & \text{if } \frac{M}{\gcd(N,M)} \text{ even} \\ \mu_3 & \text{if } \frac{N}{\gcd(N,M)} \text{ even}\,, \end{cases} \tag{4.4}$$

where ϵ is the discrete torsion parameter of (2.3) and we used that ϵ only takes the values ± 1. Due to the fact that the γ-matrices form a *projective* representation, there may appear a phase $\delta_{\bar{k}}$ in the Möbius strip:

$$\mathrm{Tr}\left(\gamma_{\Omega\bar{k}}^{-1}\gamma_{\Omega\bar{k}}^{\top}\right) = \delta_{\bar{k}}\,\mathrm{Tr}\left(\gamma_{\bar{k}}^{2}\right). \tag{4.5}$$

Finally, we define

$$s_i = \sin(\pi\bar{k}\cdot\bar{v}_i)\,, \qquad \tilde{s}_i = \sin(2\pi\bar{k}\cdot\bar{v}_i)\,, \qquad c_i = \cos(\pi\bar{k}\cdot\bar{v}_i)\,. \tag{4.6}$$

In this notation the tadpole conditions for a general $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold read:

(a) If N and M are both odd:

$$8\tilde{s}_1\tilde{s}_2\tilde{s}_3\,\mathrm{Tr}\,\gamma_{2\bar{k},3} + \sum_{i=1}^{3} 2\tilde{s}_i\,\mathrm{Tr}\,\gamma_{2\bar{k},7_i} = \delta_{\bar{k}} 32 s_1 s_2 s_3\,. \tag{4.7}$$

(b) If N is odd and M is even:

– for odd $\bar{k}$:

$$8s_1s_2s_3\,\mathrm{Tr}\,\gamma_{\bar{k},3} + \sum_{i=1}^{3} 2s_i\,\mathrm{Tr}\,\gamma_{\bar{k},7_i} = 0\,, \tag{4.8}$$

– for even twists $2\bar{k}$:

$$8\tilde{s}_1\tilde{s}_2\tilde{s}_3\,\mathrm{Tr}\,\gamma_{2\bar{k},3} + \sum_{i=1}^{3} 2\tilde{s}_i\,\mathrm{Tr}\,\gamma_{2\bar{k},7_i} = \delta_{\bar{k}} 32(s_1s_2s_3 - \tilde{\epsilon}_1 s_1 c_2 c_3)\,, \tag{4.9}$$

where $\tilde{\epsilon}_1 = \tilde{\epsilon}_{\bar{k}_1}$.

As explained in appendix B, in order to factorize the amplitudes in the appropriate way, one must impose some restrictions on the "vector structures". In the present case, one needs:[7]

$$\mu_1{}^{3} = \mu_1{}^{7_1} = -\mu_1{}^{7_2} = -\mu_1{}^{7_3} = \tilde{\epsilon}_1. \tag{4.10}$$

(c) If N and M are both even:

Only in this case discrete torsion $\epsilon = -1$ is possible. The tadpole conditions are:

– for odd $\bar{k}$:

$$8s_1s_2s_3\,\mathrm{Tr}\,\gamma_{\bar{k},3} + \sum_{i=1}^{3} 2s_i\,\mathrm{Tr}\,\gamma_{\bar{k},7_i} = 0\,, \tag{4.11}$$

[7]Strictly speaking, these relations only follow from tadpole factorization if $M > 2$. However, these conditions are still valid in the case $M = 2$. This can be seen by similar arguments as the ones used in [35]. The same applies to the relations (4.13) below.

– for even twists $2\bar{k}$:

$$8\tilde{s}_1\tilde{s}_2\tilde{s}_3 \,\mathrm{Tr}\,\gamma_{2\bar{k},3} + \sum_{i=1}^{3} 2\tilde{s}_i \,\mathrm{Tr}\,\gamma_{2\bar{k},7_i} = \epsilon^{-k_1k_2}\delta_{\bar{k}} 32 \times \\ \times \left(s_1s_2s_3 - \sum_{i\neq j\neq k} \tilde{\epsilon}_i\beta_i s_i c_j c_k \right), \tag{4.12}$$

where $\tilde{\epsilon}_i = \tilde{\epsilon}_{\bar{k}_i}$, $\beta_i = \beta_{\bar{k}_i,\bar{k}}$.

As in the previous case, there are some conditions on the "vector structures":

$$\begin{aligned} {\mu_1}^3 &= {\mu_1}^{7_1} = -{\mu_1}^{7_2} = -{\mu_1}^{7_3} = \tilde{\epsilon}_1 \,, \\ {\mu_3}^3 &= -{\mu_3}^{7_1} = -{\mu_3}^{7_2} = {\mu_3}^{7_3} = \tilde{\epsilon}_3 \,. \end{aligned} \tag{4.13}$$

In addition the $\tilde{\epsilon}_i$ are related by

$$\tilde{\epsilon}_1\tilde{\epsilon}_2\tilde{\epsilon}_3 = \epsilon^{-MN/4} \,. \tag{4.14}$$

Let us discuss some examples:

- $\mathbb{Z}_2 \times \mathbb{Z}_2$: all tadpole conditions vanish. As in the orbifold case, the contribution from the untwisted sector vanishes because the space $\mathbb{C}^3/\Gamma$ transverse to the D3-branes is non compact and the tadpoles depend on the inverse volume. The twisted sectors $(1,0)$, $(0,1)$ and $(1,1)$ each leave one complex plane fixed (the third, the first and the second, respectively). The sum over the windings along the fixed direction leads to a dependence on the inverse volume of the fixed plane. In particular, no D7-branes are needed for this model to be consistent.

- $\mathbb{Z}_3 \times \mathbb{Z}_3$: there is one twisted sector that does not fix any plane but has only a fixed point at the origin, namely the sector twisted by $g = (1,2)$ or by its inverse $g^{-1} = (2,1)$. Tadpole cancellation implies [32, 33]:

$$-\,\mathrm{Tr}\,\gamma_{(1,2),3} + \frac{1}{3}\Big(\mathrm{Tr}\,\gamma_{(1,2),7_1} + \mathrm{Tr}\,\gamma_{(1,2),7_2} - \mathrm{Tr}\,\gamma_{(1,2),7_3} \Big) = 4\delta_{(2,1)} \,. \tag{4.15}$$

  But this condition cannot be satisfied if there is discrete torsion. The only allowed values of the discrete torsion parameter different from one are $e^{\pm 2\pi i/3}$. For these values there is a unique projective representation (see appendix A) with the following character:

$$\mathrm{Tr}\,\gamma_g = 0 \quad \forall\, g \neq e \,. \tag{4.16}$$

  ($\mathrm{Tr}\,\gamma_e$ is related to the total number of branes.) Thus, we see that the above condition can never be satisfied for this projective representation. This inconsistency agrees with the one we found in the closed string sector of the same model.

A similar analysis can be done for other orientifolds, with the result that no solution can be found for certain values of the discrete torsion. In $\mathbb{Z}_N \times \mathbb{Z}_N$ models [6], with discrete torsion $\epsilon = e^{2\pi i m/N}$ and m and N coprime, there is only one irreducible representation[8] of the discrete group with this torsion as a factor system. This means that the caracters of this representation vanish for all group elements except for the unit element, i.e. (4.16) is valid for all of these models. Following the same argument as in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case, one finds that all $\mathbb{Z}_N \times \mathbb{Z}_N$, $N > 2$, orientifolds with minimal discrete torsion (i.e. $s = N$) are inconsistent. Again, this result agrees with the restrictions found in the closed string sector.

- $\mathbb{Z}_2 \times \mathbb{Z}_4$: the tadpole conditions are:

$\bar{k}$	$\bar{k}\cdot\bar{v}$	tadpole condition
$(0,1)$	$\frac{1}{4}(0,1,-1)$	$\mathrm{Tr}\,\gamma_{(0,1),7_2} - \mathrm{Tr}\,\gamma_{(0,1),7_3} = 0$
$(0,3)$	$\frac{3}{4}(0,1,-1)$	$\mathrm{Tr}\,\gamma_{(0,3),7_2} - \mathrm{Tr}\,\gamma_{(0,3),7_3} = 0$
$(1,0)$	$\frac{1}{2}(1,-1,0)$	$\mathrm{Tr}\,\gamma_{(1,0),7_1} - \mathrm{Tr}\,\gamma_{(1,0),7_2} = 0$
$(1,1)$	$\frac{1}{4}(2,-1,-1)$	$4\,\mathrm{Tr}\,\gamma_{(1,1),3} + 2\,\mathrm{Tr}\,\gamma_{(1,1),7_1} - \sqrt{2}\,\mathrm{Tr}\,\gamma_{(1,1),7_2} - \sqrt{2}\,\mathrm{Tr}\,\gamma_{(1,1),7_3} = 0$
$(1,2)$	$\frac{1}{2}(1,0,-1)$	$\mathrm{Tr}\,\gamma_{(1,2),7_1} - \mathrm{Tr}\,\gamma_{(1,2),7_3} = 0$
$(1,3)$	$\frac{1}{4}(2,1,-3)$	$-4\,\mathrm{Tr}\,\gamma_{(1,3),3} + 2\,\mathrm{Tr}\,\gamma_{(1,3),7_1} + \sqrt{2}\,\mathrm{Tr}\,\gamma_{(1,3),7_2} - \sqrt{2}\,\mathrm{Tr}\,\gamma_{(1,3),7_3} = 0$
$(0,2)$	$\frac{1}{2}(0,1,-1)$	$2\,\mathrm{Tr}\,\gamma_{(0,2),7_2} - 2\,\mathrm{Tr}\,\gamma_{(0,2),7_3} = -16\delta_{(0,1)}(\tilde{\epsilon}_2\beta_2 - \tilde{\epsilon}_3\beta_3)$

There are two cases:

(i) If $\tilde{\epsilon}_2\beta_2 = \tilde{\epsilon}_3\beta_3$, then the Klein bottle contribution in the $(0,2)$ sector vanishes. This means that no D7-branes are needed to cancel the tadpoles. This agrees with anomaly cancellation in the 33 sector. Using $\beta_2 = \beta_{(1,2),(0,1)} = \epsilon$, $\beta_3 = \beta_{(1,0),(0,1)} = \epsilon$ and eqs. (4.13), (4.14), we find that $\tilde{\epsilon}_2\beta_2 = \tilde{\epsilon}_3\beta_3$ implies ${\mu_1}^3{\mu_3}^3 = {\mu_3}^3$. Therefore this case is characterized by ${\mu_1}^3 = +1$.

(ii) If $\tilde{\epsilon}_2\beta_2 = -\tilde{\epsilon}_3\beta_3$, then D7-branes are present. In the case with discrete torsion $\epsilon = -1$, a minimal choice to satisfy the tadpole conditions consists in a set of D7_2-branes. Then all the traces vanish, except for the $(0,2)$ sector, as can be seen from (A.19). With the minimal choice, one has

$$\mathrm{Tr}\,\gamma_{(0,2),7_2} = 16\,\delta_{(0,1)}\tilde{\epsilon}_3\beta_3 = -16\,\delta_{(0,1)}{\mu_3}^3\,, \tag{4.17}$$

where we used (4.13) and $\beta_3 = \beta_{(1,0),(0,1)} = -1$. The relation $\tilde{\epsilon}_2\beta_2 = -\tilde{\epsilon}_3\beta_3$ implies ${\mu_1}^3{\mu_3}^3 = -{\mu_3}^3$ and thus ${\mu_1}^3 = -1$. From (4.13) we then find that this case is characterized by ${\mu_1}^3 = -{\mu_1}^{7_2} = -1$.

[8] In general there are N^2/s^2 different irreducible representations. If, however, $\gcd(N,m) = 1$, then the minimal positive integer s, such that $\epsilon^s = 1$, is $s = N$.

JHEP07(2000)040

4.3 Open string spectrum

An element g of the orbifold group Γ and the world-sheet parity Ω act on the Chan-Paton matrices λ as

$$g:\ \lambda \longrightarrow \gamma_g \lambda \gamma_g^{-1}\,, \qquad \Omega:\ \lambda \longrightarrow \gamma_\Omega \lambda^\top \gamma_\Omega^{-1}\,. \tag{4.18}$$

The open string spectrum, i.e. gauge fields $\lambda^{(0)}$ and matter fields $\lambda^{(i)}$, is obtained by taking the solutions to the projections:

$$\gamma_g^{-1}\lambda\gamma_g = r(g)\lambda\,, \qquad \gamma_\Omega^{-1}\lambda^\top\gamma_\Omega = r(\Omega)\lambda\,, \tag{4.19}$$

where $r(g)$ (resp. $r(\Omega)$) is the matrix that represents the action of g (resp. Ω) on $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$.

The relation $\Omega^2 = e$ gives a restriction on the matrix γ_Ω:

$$\gamma_\Omega \left(\gamma_\Omega^{-1}\right)^\top = c\,\mathbb{1}\,, \tag{4.20}$$

where the constant c can only take the values ± 1, i.e. γ_Ω is either symmetric or antisymmetric. A further condition follows from $\Omega g \Omega = g$, which, using (4.18), translates to:

$$\gamma_\Omega \left(\gamma_g^{-1}\right)^\top \left(\gamma_\Omega^{-1}\right)^\top = \delta_g\, \gamma_g\,, \tag{4.21}$$

where the phase δ_g appears because the representation is only projective. More precisely, in the notation of appendix A, $\delta_g = c\,\beta_{\Omega,g}$. This can be seen as follows. Because of the special action of Ω on λ, eq. (4.18), the matrices representing Ωg and $g\Omega$ are given by

$$\gamma_{\Omega g} = \alpha_{\Omega,g}^{-1}\,\gamma_\Omega \left(\gamma_g^{-1}\right)^\top\,, \qquad \gamma_{g\Omega} = \alpha_{g,\Omega}^{-1}\,\gamma_g\gamma_\Omega\,. \tag{4.22}$$

From $\Omega g = g\Omega$ and (4.20), we find $\delta_g = c\,\alpha_{\Omega,g}\alpha_{g,\Omega}^{-1}$. It is easy to see that the phase δ_g coincides with the one defined in (4.5).

As we are interested in unitary projective representations, (4.21) can be transformed into

$$\gamma_\Omega \gamma_g^* \gamma_\Omega^{-1} = c\,\delta_g\,\gamma_g\,, \tag{4.23}$$

where we used $\gamma_\Omega^\top = c\,\gamma_\Omega$. The world-sheet parity Ω relates one representation with its complex conjugate. This means that the matrices γ_g of the orientifold form a *(pseudo-)real* projective representation of the orbifold group. As the complex projective representations are classified by $H^2(\Gamma, \mathbb{C}^*) = H^2(\Gamma, \mathrm{U}(1))$, the real projective representations are classified by $H^2(\Gamma, \mathbb{R}^*) = H^2(\Gamma, \mathbb{Z}_2)$. As a consequence, the discrete torsion paramter ϵ can only take the real values ± 1.

Let us make one further remark concerning the phases $\beta_{\Omega,g}$. They do not satisfy (A.5) but rather $\beta_{\Omega,gh} = (\alpha_{g,h})^2\beta_{\Omega,g}\beta_{\Omega,h}$. Moreover, one can always find an equivalent set of γ-matrices with $\beta_{\Omega,g} = 1\ \forall\, g \in \Gamma$ by defining $\hat\gamma_g = \sqrt{\beta_{\Omega,g}}\,\gamma_g$. If we choose the factor system $\alpha_{g,h}$ of the orbifold group Γ as in (A.6), then $(\alpha_{g,h})^2 = 1 \; \forall \, g, h$ and therefore the $\hat{\gamma}_g$ have the same factor system as the γ_g: $\hat{\alpha}_{g,h} = \sqrt{\beta_{\Omega,g}\beta_{\Omega,h}\beta^{-1}_{\Omega,gh}}\, \alpha_{g,h} = \alpha_{g,h}$. Thus, to fix the factor system of the orientifold completely, one has to add one relation to (A.6):

$$\begin{aligned} \alpha_{g_1^a, g_2^b} = \alpha_{g_1, g_1^a} = \alpha_{g_2, g_2^b} = 1 \,, \quad a = 1, \ldots, N \,,\ b = 1, \ldots, M \,, \\ \delta_g = c \quad \forall \, g \in \Gamma \,. \end{aligned} \tag{4.24}$$

In the following, we will mostly leave $\alpha_{g,h}$, δ_g arbitrary. But to perform calculations, we choose the factor system (4.24).

The reality condition (4.23) has an interesting consequence concerning the classification of $\mathbb{Z}_N$ orientifolds. Note, that for a complex projective representation of $\mathbb{Z}_N$ the relation

$$\gamma_g^N = \tilde{c}\, \mathbb{1} \,, \qquad \tilde{c} \in \mathbb{C}^* \,, \tag{4.25}$$

can always be brought to the form

$$\hat{\gamma}_g^N = \mathbb{1} \,, \tag{4.26}$$

defining $\hat{\gamma}_g = \tilde{c}^{-1/N} \gamma_g$. For real projective representations, i.e. $\tilde{c} \in \mathbb{R}^*$, this is only possible if N is odd or if $\tilde{c} > 0$. If N is even, two inequivalent projective representations arise:

$$\gamma_g^N = \pm \mathbb{1} \,. \tag{4.27}$$

These are the cases with and without vector structure [34].

If $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$ (with N, M both even), then four cases have to be distinguished: $\gamma_{g_1}^N = \pm\mathbb{1}$, $\gamma_{g_2}^M = \pm\mathbb{1}$. Only if $\gamma_{g_1}^N = \gamma_{g_2}^M = \mathbb{1}$ and $\gamma_{g_1}\gamma_{g_2} = \gamma_{g_2}\gamma_{g_1}$, does the gauge bundle of the orientifold have vector structure in the sense of [34, 13]. Thus, orientifolds with discrete torsion can never have vector structure. However, the different boundary conditions for the gauge bundle of orientifolds with discrete torsion — let us denote them by $(++)$, $(+-)$, $(-+)$ and $(--)$ — lead, in general, to non-equivalent models. In particular, the gauge group and the spectrum are not the same in all four cases.

As explained in appendix A, there are two types of real projective representations: irreducible real projective representations and combinations of pairs of conjugate irreducible complex representations. Note, that if $\tilde{\gamma}$ is a complex projective representation of the discrete group Γ, so is $\tilde{\gamma}^* = (\tilde{\gamma}^\top)^{-1}$. The orientifold projection tells us that one has to take these two representations together, whenever $\tilde{\gamma} \neq \tilde{\gamma}^*$. In terms of matrices this means:

$$\gamma_g = \begin{pmatrix} \tilde{\gamma}_g & 0 \\ 0 & c\, \delta_g^{-1} (\tilde{\gamma}_g^\top)^{-1} \end{pmatrix} \quad \forall \, g \in \Gamma \,, \tag{4.28}$$

where $c = \pm 1$ and δ_g is a phase. It can be verified that γ_g satisfies (4.21) if γ_Ω is of the form

$$\gamma(\Omega) = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \otimes \mathbb{1}_n \,, \tag{4.29}$$

where n is the dimension of $\tilde{\gamma}$. The fact that γ_Ω can always be chosen to be of this form is a consequence of (4.20).

Now, let us see the restrictions that arise from the multiplication of two group elements $g, h \in \Gamma$:

$$\gamma_g \gamma_h = \begin{pmatrix} \tilde{\gamma}_g \tilde{\gamma}_h & 0 \\ 0 & \delta_g^{-1} \delta_h^{-1} (\tilde{\gamma}_g^\top)^{-1} (\tilde{\gamma}_h^\top)^{-1} \end{pmatrix} . \tag{4.30}$$

From the relation $\gamma_g \gamma_h = \alpha_{g,h} \gamma_{gh}$, eq. (3.6), one finds

$$\gamma_g \gamma_h = \alpha_{g,h} \begin{pmatrix} \tilde{\gamma}_{gh} & 0 \\ 0 & \delta_g^{-1} \delta_h^{-1} \alpha_{g,h}^{-2} (\tilde{\gamma}_{gh}^\top)^{-1} \end{pmatrix} = \alpha_{g,h} \, \gamma_{gh} \,, \tag{4.31}$$

which gives the relation between the discrete torsion $\alpha_{g,h}$ and the orientifold phase δ_g:

$$\delta_g \, \delta_h \, \alpha_{g,h}^2 = c \, \delta_{gh} \,. \tag{4.32}$$

This condition comes from the fact that two projective representation can be related if they belong to the same equivalence class of factor systems (see appendix A or [19, 20]). The projective representation $\tilde{\gamma}_g$ has the factor system $\alpha_{g,h}$ and $(\tilde{\gamma}_g^\top)^{-1}$ the inverse one, $\alpha_{g,h}^{-1}$. They are in the same equivalence class if there exist numbers ρ_g, such that $(\rho_g \rho_h / \rho_{gh}) \, \alpha_{g,h} = \alpha_{g,h}^{-1}$, which is just the above condition.

Interchanging g and h in (4.32), one finds $(\delta_h \delta_g / c \delta_{hg}) \alpha_{h,g} = \alpha_{h,g}^{-1}$. The quotient of these two relations gives the following consistency condition: $(\alpha_{h,g} \alpha_{g,h}^{-1})^2 = 1$. This tells us that the only values of discrete torsion compatible with the orientifold projection are

$$\beta_{g,h} = \alpha_{g,h} \alpha_{h,g}^{-1} = \pm 1 \,. \tag{4.33}$$

This result coincides with the consistency condition that we found in the closed string sector of orientifolds with discrete torsion.

According to the results of appendix A, the matrix γ_g (whith $g = g_1^a g_2^b$) of a general real projective representation with discrete torsion $\epsilon = -1$ and with $\eta_{1/2} = 0, 1$, such that $(\gamma_{g_1})^N = (-1)^{\eta_1} \mathbb{1}$, $(\gamma_{g_2})^M = (-1)^{\eta_2} \mathbb{1}$, is of the form

$$\bigoplus_{k,l} \left(\left(\omega_{2N}^{2k+\eta_1} \gamma_{g_1} \right)^a \left(\omega_{2M}^{2l+\eta_2} \gamma_{g_2} \right)^b \oplus \left(\omega_{2N}^{2N-2k-\eta_1} \gamma_{g_1} \right)^a \left(\omega_{2M}^{2M-2l-\eta_2} \gamma_{g_2} \right)^b \right) \otimes \mathbb{1}_{n_{kl}} \,, \tag{4.34}$$

where

$$\gamma_{g_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_{g_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.35}$$

as given in (A.11), and $k = 0, \ldots, N/2 - 1$, $l = 0, \ldots, M/2 - 1$. Here, we chose the factor system (A.6), which implies $(\gamma_{g_1})^a (\gamma_{g_2})^b = \gamma_{g_1^a g_2^b}$.

Let us now analyze some examples. Start with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case with $(++)$ boundary condition, i.e. $\gamma_{g_1}^2 = \gamma_{g_2}^2 = \mathbb{1}$. Discrete torsion is possible for this model, because $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. The case without discrete torsion has four irreducible representations, all of them are one-dimensional and real. The case with discrete torsion -1 has a unique irreducible representation, that can be taken to be real. A general representation, with $(++)$ boundary condition, is of the form

$$\begin{aligned} \gamma_e &= \mathbb{1}_2 \otimes \mathbb{1}_n \,, \qquad & \gamma_{g_1} &= \sigma_3 \otimes \mathbb{1}_n \,, \\ \gamma_{g_2} &= \sigma_1 \otimes \mathbb{1}_n \,, \qquad & \gamma_{g_1 g_2} &= i\sigma_2 \otimes \mathbb{1}_n \,, \end{aligned} \tag{4.36}$$

where n is an arbitrary parameter. If the matrix γ_Ω is symmetric, then it can be taken to be of the form

$$\gamma_\Omega = \mathbb{1}_2 \otimes \mathbb{1}_n. \tag{4.37}$$

If it is antisymmetric, we restrict ourselves to the case of even n and take

$$\gamma_\Omega = \mathbb{1}_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{1}_{n/2} \,. \tag{4.38}$$

Using (4.22), with $\alpha_{\Omega,g} = \alpha_{g,\Omega} = 1$, it is easy to check that all consistency conditions from the multiplication of group elements (like $g_1^2 = g_2^2 = e$, $\Omega g = g\Omega$, etc.) are satisfied for this choice of matrices. As we mentioned above, this choice is unique up to equivalence.

The spectrum can be determined by taking the matrices $\lambda^{(0)}, \lambda^{(i)}$ of the orbifold case (3.10),

$$\lambda^{(0)} = \mathbb{1}_2 \otimes X \,, \qquad \lambda^{(i)} = \sigma_i \otimes Z_i \,, \tag{4.39}$$

and restricting them, such that also the second condition in (4.19) is satisfied. If γ_Ω is symmetric, then X, Z_1, Z_3 are antisymmetric and Z_2 is symmetric. This corresponds to gauge group $\mathrm{SO}(n)$ with two adjoint fields, a traceless symmetric tensor and a singlet. If γ_Ω is antisymmetric, we find[9] $\mathrm{USp}(n)$ with two adjoint fields and an antisymmetric tensor. The superpotential is the one of the orbifold with the above representations:

$$W = \mathrm{Tr}\,(Z_1 Z_2 Z_3 + Z_2 Z_1 Z_3) \,. \tag{4.40}$$

The solution to the case with $(--)$ boundary condition is essentially the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model of [35] without D5-branes. In our conventions the γ-matrices are given by

$$\begin{aligned} \gamma_e &= \mathbb{1}_2 \otimes \mathbb{1}_{2n} \,, \qquad & \gamma_{g_1} &= (i\sigma_1 \oplus (-i\sigma_1)) \otimes \mathbb{1}_n \,, \\ \gamma_{g_2} &= (i\sigma_3 \oplus (-i\sigma_3)) \otimes \mathbb{1}_n \,, \qquad & \gamma_{g_1 g_2} &= i\sigma_2 \otimes \mathbb{1}_{2n} \,, \end{aligned} \tag{4.41}$$

[9] In our notation, $\mathrm{USp}(2k)$ is the unitary symplectic group of rank k.

where we exchanged γ_{g_1} and γ_{g_2} with respect to (4.34) for later convenience. If γ_Ω is symmetric, the spectrum consists of USp($2n$) gauge fields and three matter fields in the antisymmetric tensor representation. This is exactly the spectrum of the 99 sector of the model discussed in [35], if $n = 8$. If γ_Ω is antisymmetric, we find SO($2n$) with three symmetric tensors.

A similar analysis can be done for the boundary conditions $(+-)$ and $(-+)$. One finds in both cases the same spectrum as in the $(++)$ case.

To determine the spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orientifold, it is more convenient to use the shift formalism, as explained in appendix C. We take γ_Ω symmetric and boundary conditions $(-+)$. Let g_1 resp. g_2 be the generators of $\mathbb{Z}_2$ resp. $\mathbb{Z}_4$. We choose a basis where γ_{g_2} is diagonal. According to appendix C, g_2 is represented by the shift

$$V_{g_2} = \frac{1}{4}\left(0^{n_0}, 2^{n_0}, 1^{n_1}, 3^{n_1}\right)$$

and g_1 is represented by the permutation, acting on the roots ρ of the SO($2(n_0+n_1)$) lattice,

$$\Pi_{g_1}:\ \left(\tilde{\rho}_1^{(n_0)}, \tilde{\rho}_2^{(n_0)}, \tilde{\rho}_1^{(n_1)}, \tilde{\rho}_2^{(n_1)}\right) \longrightarrow \left(\tilde{\rho}_2^{(n_0)}, \tilde{\rho}_1^{(n_0)}, \tilde{\rho}_2^{(n_1)}, \tilde{\rho}_1^{(n_1)}\right),$$

where $\tilde{\rho}_i^{(n_0)} = (\rho_{(i-1)n_0+1}, \ldots, \rho_{in_0})$ and $\tilde{\rho}_i^{(n_1)} = (\rho_{2n_0+(i-1)n_1+1}, \ldots, \rho_{2n_0+in_1})$. As explained in appendix C no shift associated to g_1 is needed, whereas in the $(++)$ case we would need an additional shift $V_{g_1} = \frac{1}{4}(1^{\tilde{n}_0})$. (Note that g_1 and g_2 are exchanged with respect to the notation in appendix C.) The gauge group is determined by finding all linear combinations of roots ρ that satisfy $\rho \cdot V_{g_2} = 0$ and $\Pi_{g_1}(\rho) = \rho$. They are of the form

$$\begin{array}{l}
\left(\underline{+,+,0^{n_0-2}}, 0^{n_0}, 0^{2n_1}\right) + \left(0^{n_0}, \underline{+,+,0^{n_0-2}}, 0^{2n_1}\right), \\
\left(\underline{-,-,0^{n_0-2}}, 0^{n_0}, 0^{2n_1}\right) + \left(0^{n_0}, \underline{-,-,0^{n_0-2}}, 0^{2n_1}\right), \\
\left(\underline{+,-,0^{n_0-2}}, 0^{n_0}, 0^{2n_1}\right) + \left(0^{n_0}, \underline{+,-,0^{n_0-2}}, 0^{2n_1}\right), \\
\left(0^{2n_0}, \underline{+,0^{n_1-1}}, \underline{0^{n_1-1},+}\right) + \left(0^{2n_0}, \underline{0^{n_1-1},+}, \underline{+,0^{n_1-1}}\right), \\
\left(0^{2n_0}, \underline{-,0^{n_1-1}}, \underline{0^{n_1-1},-}\right) + \left(0^{2n_0}, \underline{0^{n_1-1},-}, \underline{-,0^{n_1-1}}\right), \\
\left(0^{2n_0}, \underline{+,-,0^{n_1-2}}, 0^{n_1}\right) + \left(0^{2n_0}, 0^{n_1}, \underline{+,-,0^{n_1-2}}\right).
\end{array}$$

Underlining means that one has to take all correlated permutations of the two terms in each line, such that they are invariant under Π_{g_1}. The first three lines give $n_0(2n_0-1) - n_0$ roots, which, together with n_0 Cartan generators, form the gauge group SO($2n_0$). The last three lines give $n_1(2n_1+1) - n_1$ roots, which, together with n_1 Cartan generators, form the gauge group USp($2n_1$). The matter fields from the first complex plane are obtained from the roots that satisfy $\rho \cdot V_{g_2} = 0$ and $\Pi_{g_1}(\rho) = -\rho$. These are just the antisymmetric combinations of the roots above that formed the gauge group. Thus, we find an adjoint field of SO($2n_0$) and an antisymmetric tensor of USp($2n_1$). For the second complex plane,

Γ	gauge group	matter (33 sector)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SO(2n)$	$2\ \text{adj} + \Box\!\Box$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$SO(2n_0) \times SO(2n_1)$	$2\,(\Box,\Box) + (\mathbb{1},\Box\!\Box) + (\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\mathbb{1})$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$SO(2n_0) \times U(n_1+n_2)$	$(\Box,\overline{\Box}) + (\Box,\Box)$ $+(\mathbb{1},\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) + (\mathbb{1},\Box\!\Box) + (\text{adj},\mathbb{1}) + (\mathbb{1},\text{adj})$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	$SO(2n_0) \times U(n_1+n_2)$	$3\,(\Box,\Box) + 2(\mathbb{1},\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) + (\mathbb{1},\overline{\Box\!\Box})$

Table 3: Open string spectrum for some non-compact orientifolds with discrete torsion $\epsilon = -1$ and boundary condition $(++)$, i.e. $\gamma_{g_1}^N = \mathbb{1}$, $\gamma_{g_2}^M = \mathbb{1}$.

Γ	gauge group	matter (33 sector)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SO(2n)$	$2\ \text{adj} + \Box\!\Box$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$U(n_0+n_1)$	$\text{adj} + \begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 2\,\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} + \Box\!\Box$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$SO(2n_1) \times U(n_0+n_2)$	$(\Box,\overline{\Box}) + (\Box,\Box)$ $+(\mathbb{1},\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) + (\mathbb{1},\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (\Box\!\Box,\mathbb{1}) + (\mathbb{1},\text{adj})$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	$SO(2n_1) \times U(n_0+n_2)$	$3\,(\Box,\overline{\Box}) + 2\,(\mathbb{1},\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (\mathbb{1},\Box\!\Box)$

Table 4: Open string spectrum for some non-compact orientifolds with discrete torsion $\epsilon = -1$ and boundary condition $(+-)$, i.e. $\gamma_{g_1}^N = \mathbb{1}$, $\gamma_{g_2}^M = -\mathbb{1}$.

one has the condition $\rho \cdot V_{g_2} = 1/4$ and $\Pi_{g_1}(\rho) = -\rho$. The corresponding roots are

$$
\begin{aligned}
&\big(\underline{+,0^{n_0-1}},0^{n_0},\underline{+,0^{n_1-1}},0^{n_1}\big) - \big(0^{n_0},\underline{+,0^{n_0-1}},0^{n_1},\underline{+,0^{n_1-1}}\big)\,,\\
&\big(0^{n_0},\underline{-,0^{n_0-1}},\underline{-,0^{n_1-1}},0^{n_1}\big) - \big(\underline{-,0^{n_0-1}},0^{n_0},0^{n_1},\underline{-,0^{n_1-1}}\big)\,,\\
&\big(\underline{-,0^{n_0-1}},0^{n_0},\underline{+,0^{n_1-1}},0^{n_1}\big) - \big(0^{n_0},\underline{-,0^{n_0-1}},0^{n_1},\underline{+,0^{n_1-1}}\big)\,,\\
&\big(\underline{+,0^{n_0-1}},0^{n_0},0^{n_1},\underline{-,0^{n_1-1}}\big) - \big(0^{n_0},\underline{+,0^{n_0-1}},\underline{-,0^{n_1-1}},0^{n_1}\big)\,.
\end{aligned}
$$

They form a matter field transforming in the bifundamental $(\Box,\Box)$ representation of the gauge group. The matter fields from the third complex plane correspond to roots that satisfy $\rho \cdot V_{g_2} = -1/4$ and $\Pi_{g_1}(\rho) = \rho$. Again, one finds $4n_0n_1$ roots, giving a second bifundamental.

A similar analysis can be performed for the other boundary conditions, $(+-)$, $(++)$, $(--)$. The result is shown in tables 3–6. One finds that the $(+-)$ model and the $(--)$ model have non-abelian gauge anomalies in the 33 sector. In the language of the previous subsection, these two boundary conditions correspond to ${\mu_1}^3 = -1$. There, we saw that precisely these models need D7-branes to cancel the tadpoles. A minimal choice consists in a set of $D7_2$-branes satisfying $\operatorname{Tr}\gamma_{(0,2),7_2} = -16\,{\mu_3}^3\delta_{(0,1)}$. Let us consider the $(--)$ model with γ_Ω symmetric. According to (4.24), this gives $\delta_g = 1$. From (4.13), we have ${\mu_1}^{7_2} = -{\mu_1}^3$ and ${\mu_3}^{7_2} = -{\mu_3}^3$.

Γ	gauge group	matter (33 sector)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathrm{SO}(2n)$	$2\ \mathrm{adj} + \Box\!\Box$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathrm{SO}(2n_0) \times \mathrm{USp}(2n_1)$	$(\mathrm{adj}, \mathbb{1}) + (\mathbb{1}, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 2\,(\Box, \Box)$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1 + n_2)$	$(\Box, \overline{\Box}) + (\Box, \Box) + (\mathbb{1}, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (\mathbb{1}, \overline{\Box\!\Box})$ $+(\mathrm{adj}, \mathbb{1}) + (\mathbb{1}, \mathrm{adj})$
$\mathbb{Z}_2 \times \mathbb{Z}'_6$	$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1 + n_2)$	$3\,(\Box, \Box) + 2, (\mathbb{1}, \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) + (\mathbb{1}, \overline{\Box\!\Box})$

Table 5: Open string spectrum for some non-compact orientifolds with discrete torsion $\epsilon = -1$ and boundary condition $(-+)$, i.e. $\gamma^N_{g_1} = -\mathbb{1}$, $\gamma^M_{g_2} = \mathbb{1}$.

Γ	gauge group	matter (33 sector)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathrm{USp}(2n)$	$3\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathrm{U}(n_0 + n_1)$	$\mathrm{adj} + 2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} + \overline{\Box\!\Box}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathrm{USp}(2n_1) \times \mathrm{U}(n_0 + n_2)$	$(\Box, \overline{\Box}) + (\Box, \Box)$ $+(\mathbb{1}, \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) + (\mathbb{1}, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \mathbb{1}) + (\mathbb{1}, \mathrm{adj})$
$\mathbb{Z}_2 \times \mathbb{Z}'_6$	$\mathrm{USp}(2n_1) \times \mathrm{U}(n_0 + n_2)$	$3\,(\Box, \overline{\Box}) + 3\,(\mathbb{1}, \begin{smallmatrix}\Box\\\Box\end{smallmatrix})$

Table 6: Open string spectrum for some non-compact orientifolds with discrete torsion $\epsilon = -1$ and boundary condition $(--)$, i.e. $\gamma^N_{g_1} = -\mathbb{1}$, $\gamma^M_{g_2} = -\mathbb{1}$.

Thus the theory on the $\mathrm{D7}_2$-branes has $(++)$ boundary condition. As explained in [36, 35], $\gamma_{\Omega_{7_2}}$ has to be antisymmetric.[10] In table 3 the general solution for the gauge theory on a set of D-branes at a $\mathbb{Z}_2 \times \mathbb{Z}_4$ singularity is shown for the case of symmetric γ_Ω. Knowing that changing γ_Ω from symmetric to antisymmetric exchanges *SO*-factors with USp-factors, we find that the gauge theory on the $\mathrm{D7}_2$-branes is $\mathrm{USp}(2m_0) \times \mathrm{USp}(2m_1)$. In order to cancel the tadpoles, we need $2(2m_0 - 2m_1) = 16$. The 37_2 sector gives matter fields transforming in the representations $(\Box; \Box, \mathbb{1})$ and $(\overline{\Box}; \mathbb{1}, \Box)$ under the total gauge group $\mathrm{U}(n_0 + n_1) \times \mathrm{USp}(2m_0) \times \mathrm{USp}(2m_1)$. The total anomaly is thus given by $2m_0 - 2m_1 - 8$. We see that the condition of anomaly freedom is equivalent to the condition of vanishing tadpoles.

In the same way the $\mathbb{Z}_2 \times \mathbb{Z}_6$ and the $\mathbb{Z}_2 \times \mathbb{Z}'_6$ orientifold can be constructed. The result is shown in tables 3–6. Again, some of these models have non-abelian gauge anomalies in the 33 sector. It can be seen that the same models require D7-branes for tadpole cancellation.

In general, one finds the following gauge groups for $\mathbb{Z}_2 \times \mathbb{Z}_N$ orientifolds with discrete torsion (if γ_Ω is symmetric):

[10] If, instead of the standard Ω-projection discussed by Gimon and Polchinski [36], we use the alternative projection proposed by Dabholkar and Park in $D = 6$ [37], $\gamma_{\Omega,7_2}$ and $\gamma_{\Omega,3}$ have the same symmetry.

- boundary condition $(++)$:

$$
\begin{aligned}
N = 4k+2 \ &: \ G = \mathrm{SO}(2n_0) \times \prod_{i=1}^{k} \mathrm{U}(n_i + n_{N/2-i})\,, \\
N = 4k \ &: \quad\ \ G = \mathrm{SO}(2n_0) \times \mathrm{SO}(2n_k) \times \prod_{i=1}^{k-1} \mathrm{U}(n_i + n_{N/2-i})\,.
\end{aligned} \tag{4.42}
$$

- boundary condition $(+-)$:

$$
\begin{aligned}
N = 4k+2 \ &: \ G = \mathrm{SO}(2n_k) \times \prod_{i=0}^{k-1} \mathrm{U}(n_i + n_{N/2-1-i})\,, \\
N = 4k \ &: \quad\ \ G = \prod_{i=0}^{k-1} \mathrm{U}(n_i + n_{N/2-1-i})\,.
\end{aligned} \tag{4.43}
$$

- boundary condition $(-+)$:

$$
\begin{aligned}
N = 4k+2 \ &: \ G = \mathrm{SO}(2n_0) \times \prod_{i=1}^{k} \mathrm{U}(n_i + n_{N/2-i})\,, \\
N = 4k \ &: \quad\ \ G = \mathrm{SO}(2n_0) \times \mathrm{USp}(2n_k) \times \prod_{i=1}^{k-1} \mathrm{U}(n_i + n_{N/2-i})\,.
\end{aligned} \tag{4.44}
$$

- boundary condition $(--)$:

$$
\begin{aligned}
N = 4k+2 \ &: \ G = \mathrm{USp}(2n_k) \times \prod_{i=0}^{k-1} \mathrm{U}(n_i + n_{N/2-1-i})\,, \\
N = 4k \ &: \quad\ \ G = \prod_{i=0}^{k-1} \mathrm{U}(n_i + n_{N/2-1-i})\,.
\end{aligned} \tag{4.45}
$$

4.4 Resolution of the singularities

Some of the deformations that are available in the orbifold case are absent in the orientifold case.

Take the easiest example: $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the orbifold case one can find three possible deformations [5]:

$$
\Delta W = \sum_{i=1}^{3} \zeta_i \,\mathrm{Tr}\, Z_i\,. \tag{4.46}
$$

For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $(++)$ boundary condition, only one of the deformations survives the orientifold projection. Because of the symmetry of the Z_1, Z_3 (they are antisymmetric) and Z_2 (symmetric) matrices, one has:

$$
\zeta_1 \,\mathrm{Tr}\, Z_1 = 0\,, \qquad \zeta_3 \,\mathrm{Tr}\, Z_3 = 0\,. \tag{4.47}
$$

This indicates that the orientifold can only be deformed to the $\mathbb{Z}_2$ singularity. The additional deformation leading to the conifold is frozen in the orientifold case. Something similar happened in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold without discrete torsion [30].

5. Conclusions

We have seen that only real values of the discrete torsion parameter ϵ are allowed for orientifold models, in contrast to the orbifold case, where ϵ can take complex values. This condition can be understood from several viewpoints. In the open string sector this is related to the fact that the γ-matrices form *real* projective representations. These are classified by $H^2(\Gamma, \mathbb{R}^*) = H^2(\Gamma, \mathbb{Z}_2)$, in a similar manner as complex projective representations are characterized by $H^2(\Gamma, \mathbb{C}^*) = H^2(\Gamma, \mathrm{U}(1))$. As a consequence, only $\epsilon = \pm 1$ is allowed for orientifolds. In the closed string spectrum, the condition of real ϵ is related to the matching between left- and right-moving degrees of freedom. In general, this matching is impossible if the Hodge numbers $h^{1,2}$ and $h^{2,1}$ from one twisted sector are different. One finds that $h^{1,2} = h^{2,1}$ is only guaranteed in each twisted sector if $\epsilon = \pm 1$. Finally, one finds an inconsistency in the tadpole cancellation conditions for non-real ϵ. The characters of the projective representation γ must have a precise value to cancel the Klein bottle contribution. This conditions cannot be satisfied if arbitrary values of the discrete torsion are allowed.

For the $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold non-trivial discrete torsion is only possible if N and M are both even. In this case, there are four non-equivalent orientifold 0models. This classification is based on the possibility of imposing different boundary conditions ("vector structures") on the γ-matrices. The tadpole conditions are different for each case. Some of them require D7-branes for consistency. This condition is equivalent to the requirement that non-abelian gauge anomalies be absent.

We have also analyzed the resolution of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity in both cases, orbifold and orientifold. In the orbifold case, one can get a $\mathbb{Z}_2$ singularity and a conifold by the deformations of the F-flatness conditions. In the orientifold only the $\mathbb{Z}_2$ singularity can be obtained.

This discussion can be generalized to compact orientifolds with discrete torsion [17]. The S-dual heterotic models are related to higher level Kac-Moody algebras. In particular, as the only non-trivial discrete torsion is $\epsilon = -1$, we will find heterotic duals realized at Kac-Moody level 2 [38, 39, 40].

Acknowledgments

It is a pleasure to thank Angel Uranga and Luis Ibáñez for many helpful ideas and comments on the manuscript. The work of M.K. is supported by a TMR network of the European Union, ref. FMRX-CT96-0090. The work of R.R. is supported by the MEC through a FPU Grant.

A. Projective representations

The matrices that represent the action of the orbifold group on the Chan-Paton indices of open strings form projective representations. In this appendix, we review some useful facts about the latter (for an introduction to this subject see [19, 20]). Let Γ be a finite group. A projective representation is a mapping $\Gamma \to \mathrm{GL}(n,\mathbb{K})$, which associates to each element $g \in \Gamma$ a matrix $\gamma_g \in \mathrm{GL}(n,\mathbb{K})$ that satisfies the conditions[11]

$$\gamma_e = \mathbb{1}\,, \qquad \gamma_g \gamma_h = \alpha_{g,h} \gamma_{gh} \quad \forall\, g,h \in \Gamma\,, \tag{A.1}$$

where e is the neutral element of Γ and $\alpha_{g,h} \in \mathbb{K}^*$ are arbitrary non-zero numbers. These numbers are called the *factor system* of the projective representation γ. Using associativity, one immediately obtains

$$\alpha_{g,e} = \alpha_{e,g} = 1\,, \qquad \alpha_{g,hk}\alpha_{h,k} = \alpha_{g,h}\alpha_{gh,k} \quad \forall\, g,h,k \in \Gamma\,. \tag{A.2}$$

A map $\alpha : \Gamma \times \Gamma \to \mathbb{K}^*$, $(g,h) \mapsto \alpha_{g,h}$, with the above properties is called a *cocycle*. The set of all cocycles is denoted by $Z^2(\Gamma,\mathbb{K}^*)$. A projective representation $\hat{\gamma}$ is considered equivalent to γ if it is obtained by substituting $\hat{\gamma}_g = \rho_g \gamma_g$, with $\rho_g \in \mathbb{K}^*$. The cocycles are then related by

$$\hat{\alpha}_{g,h} = \alpha_{g,h}\, \rho_g \rho_h \rho_{gh}^{-1}\,. \tag{A.3}$$

This motivates the following definition. Given a map $\rho : \Gamma \to \mathbb{K}^*$, $g \mapsto \rho_g$, with $\rho_e = 1$, one defines the associated *coboundary* by $\delta\rho : \Gamma \times \Gamma \to \mathbb{K}^*$, $(g,h) \mapsto \rho_g \rho_h \rho_{gh}^{-1}$. The set of all coboundaries, $B^2(\Gamma,\mathbb{K}^*)$, is a subset of $Z^2(\Gamma,\mathbb{K}^*)$. Two cocycles are equivalent if they differ only by a coboundary. We see that the non-equivalent projective representations are characterized by the elements of $H^2(\Gamma,\mathbb{K}^*) = Z^2(\Gamma,\mathbb{K}^*)/B^2(\Gamma,\mathbb{K}^*)$.

If Γ is abelian, one finds from (A.1) that

$$\gamma_g \gamma_h = \beta_{g,h} \gamma_h \gamma_g\,, \quad \text{where } \beta_{g,h} = \alpha_{g,h}\alpha_{h,g}^{-1}\,. \tag{A.4}$$

The $\beta_{g,h}$ only depend on the equivalence class of the $\alpha_{g,h}$. Furthermore they satisfy

$$\beta_{g,g} = 1\,, \qquad \beta_{g,h} = \beta_{h,g}^{-1}\,, \qquad \beta_{g,hk} = \beta_{g,h}\beta_{g,k} \quad \forall\, g,h,k \in \Gamma\,. \tag{A.5}$$

It is clear that to each element $[\alpha]$ of $H^2(\Gamma,\mathbb{K}^*)$ there corresponds a unique cocycle[12] β, as given in (A.4). On the other hand to each β there corresponds a unique (up to equivalence) factor system $\alpha_{g,h} = \sqrt{\beta_{g,h}}$. As a consequence, the projective representations of an abelian finite group Γ can be characterized either by (A.1), (A.2)

[11] Equivalently a projective representations can be defined as a homomorphism $\Gamma \to \mathrm{PGL}(n,\mathbb{K})$.

[12] Note that β is a cocycle because it satisfies (A.2), but it is not a factor system because its definition differs from (A.1).

or by the first eq. of (A.4) and the three eqs. of (A.5). Both descriptions are equivalent up to a transformation (A.3). Sometimes it is useful to fix as many of the $\alpha_{g,h}$ as possible without putting any restriction on the equivalence class $[\alpha]$. A convenient choice is:

$$\alpha_{g_1^a,g_2^b} = \alpha_{g_1,g_1^a} = \alpha_{g_2,g_2^b} = 1\,, \quad a = 1,\ldots,N\,,\ b = 1,\ldots,M\,. \tag{A.6}$$

This corresponds to choosing a set of matrices that satisfies

$$(\gamma_{g_1})^a(\gamma_{g_2})^b = \gamma_{g_1^a g_2^b}\,. \tag{A.7}$$

For physical reasons we restrict ourselves to unitary projective representations over the complex or real numbers, i.e. $\gamma_g\gamma_g^\dagger = \mathbb{1}$ and $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. One can prove that[13]

$$H^2(\Gamma,\mathbb{C}^*) = H^2(\Gamma,\mathrm{U}(1))\,, \qquad H^2(\Gamma,\mathbb{R}^*) = H^2(\Gamma,\mathbb{Z}_2)\,. \tag{A.8}$$

In this article we will be mainly interested in the case where $\Gamma = \mathbb{Z}_N\times\mathbb{Z}_M$. Let g_1 and g_2 be the generators of $\mathbb{Z}_N$ and $\mathbb{Z}_M$ respectively and denote the element $g_1^a g_2^b \in \Gamma$ by (a,b). The form of the β cocycles is completely fixed by the conditions (A.5). If $\mathbb{K} = \mathbb{C}$, there are $p = \gcd(N,M)$ non-equivalent cocycles, given by

$$\beta_{(a,b),(a',b')} = \omega_p^{m(ab'-ba')}\,, \qquad \text{with}\ \ \omega_p = e^{2\pi i/p}\,,\ m = 1,\ldots,p\,. \tag{A.9}$$

The different projective representations are therefore determined by the parameter $\epsilon = \omega_p^m$. Choosing the factor system (A.6), one finds a simple relation for the product of two γ-matrices:

$$\gamma_{(a,b)}\gamma_{(c,d)} = \epsilon^{-bc}\gamma_{(a+c,b+d)}\,. \tag{A.10}$$

The possibility of having $\epsilon \neq 1$ is the open string analogue of the discrete torsion in closed string orbifolds discussed in [1, 2, 3]. The open string models with discrete torsion are distinguished by the integer $s = p/\gcd(m,p)$ (this is the smallest non-zero number such that $\epsilon^s = 1$). According to [8], any irreducible projective representation $\mathcal{R}^{(s)}_{\mathrm{irr}}$ is s-dimensional and (up to projective equivalence) of the form

$$\gamma_{g_1} = \mathrm{diag}\left(1,\epsilon^{-1},\epsilon^{-2},\ldots,\epsilon^{-(s-1)}\right), \qquad \gamma_{g_2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & 0 & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}. \tag{A.11}$$

All the the irreducible projective representations $\mathcal{R}^{(s)}_{\mathrm{irr},k,l}$ that are *linearly* non-equivalent (i.e. they belong to different factor systems but, of course, are projectively equivalent) can be obtained by multiplying the matrices in (A.11) by phases:

$$\hat\gamma_{g_1} = \omega_N^k\gamma_{g_1}\,, \quad k = 0,\ldots,\frac{N}{s}-1\,, \qquad \hat\gamma_{g_2} = \omega_M^l\gamma_{g_1}\,, \quad l = 0,\ldots,\frac{M}{s}-1\,. \tag{A.12}$$

[13]For the cocycles related to unitary projective representations this is obvious. In general, this is a consequence of [19, proposition 2.3.10 and lemma 2.3.19].

A general projective representation $\mathcal{R}^{(s)}$ is a direct sum of irreducible blocks $\mathcal{R}^{(s)}_{\mathrm{irr},k,l}$:

$$\mathcal{R}^{(s)} = \bigoplus_{k,l} n_{kl} \mathcal{R}^{(s)}_{\mathrm{irr},k,l} \,. \tag{A.13}$$

This representation has dimension $s \sum_{k,l} n_{kl}$. The regular represention, of dimension $|\Gamma| = NM$, is obtained by setting $n_{kl} = s \; \forall\, k,l$. Let $\gamma_g^{(s)}$ denote the matrix associated to the group element g in the general projective representation $\mathcal{R}^{(s)}$. The *character* of g in $\mathcal{R}^{(s)}$ is defined to be the trace of this matrix. Denoting again $g = g_1^a g_2^b$ by (a,b) we find

$$\chi^{(\mathcal{R}^{(s)})}(g) \equiv \mathrm{Tr}\, \gamma_g^{(s)} = \begin{cases} 0 & \text{if } (a,b) \notin s\mathbb{Z} \times s\mathbb{Z} \\ s \sum\limits_{k,l} n_{k,l} \omega_N^{ka} \omega_M^{lb} & \text{if } (a,b) \in s\mathbb{Z} \times s\mathbb{Z} \,. \end{cases} \tag{A.14}$$

The solutions for $\mathbb{K} = \mathbb{R}$ can be obtained by restricting β to be real. The non-equivalent cocycles correspond to setting $m = p/2$ (if p is even) or $m = p$ in (A.9). We see that discrete torsion is only possible if p is even. The only non-trivial cocycle in this case is $\beta_{(a,b),(a',b')} = (-1)^{ab'-ba'}$. The irreducible projective representations are again of the form (A.11), with $\epsilon = -1$ and $s = 2$. However, there are four different complex projective representations which are not equivalent over the real numbers and which have a real factor system:[14]

$$\begin{aligned} &\text{(i) } \gamma_{g_1} \,, && \gamma_{g_2} \,, \\ &\text{(ii) } \omega_{2N}\gamma_{g_1} \,, && \gamma_{g_2} \,, \\ &\text{(iii) } \gamma_{g_1} \,, && \omega_{2M}\gamma_{g_2} \,, \\ &\text{(iv) } \omega_{2N}\gamma_{g_1} \,, && \omega_{2M}\gamma_{g_2} \,, \end{aligned} \tag{A.15}$$

where $\gamma_{g_{1/2}}$ are given in (A.11). In general, an irreducible complex projective representation is of the form $\tilde{\gamma}_{g_{1/2}} = \rho_{1/2}\gamma_{g_{1/2}}$, with $\rho_{1/2} \in \mathbb{C}^*$. The condition that the factor system be real implies $(\tilde{\gamma}_{g_1})^N = c_1\, \mathbb{1}$, $(\tilde{\gamma}_{g_2})^M = c_2\, \mathbb{1}$, with $c_{1/2} \in \mathbb{R}^*$. Therefore, $\rho_1^N = \pm 1$ and $\rho_2^M = \pm 1$. These are the four possibilities of (A.15). Defining $\eta = (\eta_1, \eta_2)$, with $\eta_i \in \{0,1\}$ by

$$(\gamma_{g_1})^N = (-1)^{\eta_1}\, \mathbb{1} \,, \qquad (\gamma_{g_2})^M = (-1)^{\eta_2}\, \mathbb{1} \,, \tag{A.16}$$

we can give the form of the linearly non-equivalent real projective representations:

$$\mathcal{R}^{\mathrm{real}}_{\mathrm{irr},k,l,\eta} = \begin{cases} \mathcal{R}^{(2)}_{\mathrm{irr},k,l,\eta} & \text{if } k = l = \eta_1 = \eta_2 = 0 \\ \mathcal{R}^{(2)}_{\mathrm{irr},k,l,\eta} \oplus (\mathcal{R}^{(2)}_{\mathrm{irr},k,l,\eta})^* & \text{else} \,, \end{cases} \tag{A.17}$$

where $\mathcal{R}^{(2)}_{\mathrm{irr},k,l}$ is obtained from (A.11) by multiplying γ_{g_1} with phases $\omega_N^k \omega_{2N}^{\eta_1}$ and γ_{g_2} with phases $\omega_M^l \omega_{2M}^{\eta_2}$ as in (A.12) and (A.15). These projective representations are either two dimensional (first line of (A.17)) or four dimensional (second line of (A.17)).

[14]We used that M and N are even, which is true because $\gcd(M,N)$ is even.

Again, a general projective representation $\mathcal{R}_\eta^{\rm real}$ is a direct sum of irreducible blocks $\mathcal{R}_{{\rm irr},k,l,\eta}^{\rm real}$:

$$\mathcal{R}_\eta^{\rm real} = \bigoplus_{k,l} n_{kl} \mathcal{R}_{{\rm irr},k,l,\eta}^{\rm real} \,. \tag{A.18}$$

This representation has dimension $2rn_{0,0} + 4\sum_{(k,l)\neq(0,0)} n_{kl}$, where $r = 2^{|\eta_1-\eta_2+\eta_1\eta_2|}$. The regular representation is obtained by setting $n_{0,0} = 2/r$ and $n_{kl} = 1\ \forall\,(k,l) \neq (0,0)$. For the character of an element $g = g_1^a g_2^b$ we find

$$\begin{aligned} \chi^{(\mathcal{R}_\eta^{\rm real})}(g) &\equiv {\rm Tr}\,\gamma_g^{\rm real} \\ &= \begin{cases} 0 & \text{if } a \text{ or } b \text{ is odd} \\ 2\,r\,n_{0,0} + 4 \displaystyle\sum_{(k,l)\neq(0,0)} n_{k,l}\,{\rm Re}\left(\omega_N^{(k+\eta_1/2)a}\omega_M^{(l+\eta_2/2)b}\right) & \text{if } a \text{ and } b \text{ are even} \\ \text{with } r=1 \text{ if } \eta_1=\eta_2=0 \text{ and } r=2 & \text{else}\,. \end{cases} \end{aligned} \tag{A.19}$$

B. Tadpoles for $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds

We sketch the calculation of the tadpoles for non-compact $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds. Basically, we follow the work of [32, 30, 23].

The set-up consists of several D3-branes at the orbifold singularity plus some D7$_i$-branes, where the i denotes the complex plane with Dirichlet conditions. We take the orientifold involution $\Omega' = \Omega(-1)^{F_L} R_1 R_2 R_3$ (other types can be considered, as in [30]).

The procedure we will follow is based on the computation of the tadpoles for $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ and then taking the non-compact limit. In order to take this limit, one must ignore the untwisted tadpoles and the contributions with an inversely proportional dependence on the volume. The dependence proportional to the volume is related to some factors from the momentum modes which become continuous in the non-compact limit.

To simplify the notation, let us define [30] $s_i = \sin(\pi\bar{k}\cdot\bar{v}_i)$, $c_i = \cos(\pi\bar{k}\cdot\bar{v}_i)$ and $\tilde{s}_i = \sin(2\pi\bar{k}\cdot\bar{v}_i)$. We will not give explicitly the volume dependence, but this dependence can be extracted from the zeros and divergences of each contribution.

The cylinder amplitude can be split into four sectors: 33, 7_i7_i, 7_i7_j and 7_i3. All the contributions can be neatly recast:

$$\mathcal{C} = \sum_{\bar{k}=(1,1)}^{(N,M)} \frac{1}{8s_1s_2s_3}\left[8s_1s_2s_3\,{\rm Tr}\,\gamma_{\bar{k},3} + \sum_{i=1}^{3} 2s_i\,{\rm Tr}\,\gamma_{\bar{k},7_i}\right]^2 . \tag{B.1}$$

Note that in the orbifold case, this is the only contribution, that has to be considered.

Because of the J operation that relates the $\bar{k}$ with the $-\bar{k}$ twisted sector, the contributions to the Klein bottle come from the untwisted and order-two twisted sectors. Three cases must be distinguished:

(a) N and M are both odd: there are no order-two twisted sectors.

(b) N is odd and M is even: there is only one order-two sector: $\bar{k}_1 = (0, M/2)$. It fixes the first complex plane. Note that in this case no discrete torsion is allowed. The Klein bottle contribution is of the form:

$$\mathcal{K} = \sum_{\bar{k}} \left(\mathcal{K}_0(\bar{k}) + \mathcal{K}_1(\bar{k})\right), \tag{B.2}$$

where $\mathcal{K}_0$ is the contribution from the untwisted sector and $\mathcal{K}_1$ is the contribution from the order-two twisted sector that fixes the first complex plane.

(c) N and M are both even: there are three order-two sectors: $\bar{k}_3 = (N/2, 0)$, $\bar{k}_1 = (0, M/2)$ and $\bar{k}_2 = (N/2, M/2)$. The Klein bottle contribution can be written as

$$\mathcal{K} = \sum_{\bar{k}} \Big(\mathcal{K}_0(\bar{k}) + \mathcal{K}_1(\bar{k}) + \mathcal{K}_2(\bar{k}) + \mathcal{K}_3(\bar{k})\Big), \tag{B.3}$$

where $\mathcal{K}_i$ is the contribution from the order-two twist that fixes the i-th complex plane. The untwisted sector contribution is of the form

$$\mathcal{K}_0(\bar{k}) = 16 \frac{2\tilde{s}_1 2\tilde{s}_2 2\tilde{s}_3}{4c_1^2 4c_2^2 4c_3^2}, \tag{B.4}$$

and the order-two twisted contributions are

$$\mathcal{K}_i(\bar{k}) = -16 \tilde{\epsilon}_i \beta_i \frac{2\tilde{s}_i}{4c_i^2}, \tag{B.5}$$

where $\tilde{\epsilon}_i$ is a sign that weights the contribution of the sector that fixes the i-th complex plane and $\beta_i = \beta_{\bar{k}_i, \bar{k}}$.

This Klein bottle contributions can be rewritten for each of the three cases:

(a) N odd, M odd:

$$\mathcal{K} = \sum_{\bar{k}=(1,1)}^{(N,M)} \frac{1}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3} [32 s_1 s_2 s_3]^2 . \tag{B.6}$$

(b) N odd, M even:

$$\mathcal{K} = \sum_{\bar{k}=(1,1)}^{(N,M/2)} \frac{1}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3} \Big[32(s_1 s_2 s_3 - \tilde{\epsilon}_1 s_1 c_2 c_3)\Big]^2 . \tag{B.7}$$

(c) N even, M even:

$$\mathcal{K} = \sum_{\bar{k}=(1,1)}^{(N/2,M/2)} \frac{1}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3} \left[32 \left(s_1 s_2 s_3 - \sum_{i \neq j \neq k} \tilde{\epsilon}_i \beta_i s_i c_j c_k\right)\right]^2 . \tag{B.8}$$

Finally, one must consider the Möbius strip contribution. For each of the four types of branes there is a contribution:

$$\mathcal{M}_3(\bar{k}) = -8 \cdot 8 s_1 s_2 s_3 \, \mathrm{Tr}\left(\gamma_{\Omega\bar{k},3}^{-1}\gamma_{\Omega\bar{k},3}^{\top}\right), \tag{B.9}$$

$$\mathcal{M}_{7_i}(\bar{k}) = 8\frac{8 s_i c_j c_k}{4c_j^2 4c_k^2} \, \mathrm{Tr}\left(\gamma_{\Omega\bar{k},7_i}^{-1}\gamma_{\Omega\bar{k},7_i}^{\top}\right). \tag{B.10}$$

Using the properties (4.21), (4.22) and (A.10) of projective representations, we find

$$\gamma_{\Omega\bar{k}}^{-1}\gamma_{\Omega\bar{k}}^{\top} = \delta_{\bar{k}}(\gamma_{\bar{k}}^2)^{\top} = \epsilon^{-k_1 k_2}\delta_{\bar{k}}(\gamma_{\bar{2k}})^{\top}, \tag{B.11}$$

where $\bar{k} = (k_1, k_2)$ and we chose a factor system such that $(\gamma_{(1,0)})^{k_1}(\gamma_{(0,1)})^{k_2} = \gamma_{\bar{k}}$.

Reordering the contributions, as we have done for the Klein bottle, one can obtain:

(*a*) N odd, M odd:

From D3-branes:

$$\mathcal{M}_3 = -2\sum_{\bar{k}=(1,1)}^{(N,M)} \frac{\delta_{\bar{k},3}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3}[32 s_1 s_2 s_3]\left[8\tilde{s}_1\tilde{s}_2\tilde{s}_3 \, \mathrm{Tr}\,\gamma_{2\bar{k},3}\right]. \tag{B.12}$$

From D7$_i$-branes:

$$\mathcal{M}_{7_i} = 2\sum_{\bar{k}=(1,1)}^{(N,M)} \frac{\delta_{\bar{k},7_i}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3}[32 s_1 s_2 s_3]\left[2\tilde{s}_i \, \mathrm{Tr}\,\gamma_{2\bar{k},7_i}\right]. \tag{B.13}$$

(*b*) N odd, M even:

From D3-branes:

$$\mathcal{M}_3 = -2\sum_{\bar{k}=(1,1)}^{(N,M/2)} \frac{\delta_{\bar{k},3}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3}\Big[32(s_1 s_2 s_3 - \tilde{\epsilon}_1 s_1 c_2 c_3)\Big]\left[8\tilde{s}_1\tilde{s}_2\tilde{s}_3 \, \mathrm{Tr}\,\gamma_{2\bar{k},3}\right]. \tag{B.14}$$

From D7$_i$-branes:

$$\mathcal{M}_{7_i} = 2\sum_{\bar{k}=(1,1)}^{(N,M/2)} \frac{\delta_{\bar{k},7_i}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3}\Big[32(s_1 s_2 s_3 - \tilde{\epsilon}_1 s_1 c_2 c_3)\Big]\left[2\tilde{s}_i \, \mathrm{Tr}\,\gamma_{2\bar{k},7_i}\right]. \tag{B.15}$$

This factorization is necessary to match the cylinder and the Klein bottle contributions. But to obtain it, we had to impose some restrictions on the "vector structures" μ_i (introduced in (4.3)) and the signs $\tilde{\epsilon}_i$:

$$\mu_1{}^3 = \mu_1{}^{7_1} = -\mu_1{}^{7_2} = -\mu_1{}^{7_3} = \tilde{\epsilon}_1. \tag{B.16}$$

Here we took $\delta_{\bar{k}} = 1 \; \forall\bar{k}$ if γ_Ω is symmetric and $\delta_{\bar{k}} = -1 \; \forall\bar{k}$ if γ_Ω is antisymmetric, as in (4.24).

(*c*) N even, M even:

From D3-branes:

$$\mathcal{M}_3 = -2 \sum_{\bar{k}=(1,1)}^{(N/2,M/2)} \frac{\epsilon^{-k_1k_2}\delta_{\bar{k},3}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3} \left[32\left(s_1s_2s_3 - \sum_{i\neq j\neq k} \tilde{\epsilon}_i\beta_i s_i c_j c_k\right)\right] \times \\ \times \left[8\tilde{s}_1\tilde{s}_2\tilde{s}_3 \,\mathrm{Tr}\,\gamma_{2\bar{k},3}\right] . \tag{B.17}$$

From D7_i-branes:

$$\mathcal{M}_{7_i} = 2 \sum_{\bar{k}=(1,1)}^{(N/2,M/2)} \frac{\epsilon^{-k_1k_2}\delta_{\bar{k},7_i}}{8\tilde{s}_1\tilde{s}_2\tilde{s}_3} \left[32\left(s_1s_2s_3 - \sum_{i\neq j\neq k} \tilde{\epsilon}_i\beta_i s_i c_j c_k\right)\right] \times \\ \times \left[2\tilde{s}_i \,\mathrm{Tr}\,\gamma_{2\bar{k},7_i}\right] . \tag{B.18}$$

Again, this factorization is necessary to match the cylinder and the Klein bottle contributions, but it is only possible if we impose some restrictions on the "vector structures" μ_i and the signs $\tilde{\epsilon}_i$:

$$\begin{aligned} {\mu_1}^3 = {\mu_1}^{7_1} = -{\mu_1}^{7_2} = -{\mu_1}^{7_3} &= \tilde{\epsilon}_1 \,, \\ {\mu_3}^3 = -{\mu_3}^{7_1} = -{\mu_3}^{7_2} = {\mu_3}^{7_3} &= \tilde{\epsilon}_3 \,, \\ \tilde{\epsilon}_1\tilde{\epsilon}_2\tilde{\epsilon}_3 &= \epsilon^{-MN/4} . \end{aligned} \tag{B.19}$$

One further condition is needed, to be able to write the total tadpole contribution $\mathcal{C} + \mathcal{M} + \mathcal{K}$ as a square:[15]

$$\delta_{\bar{k},3} = -\delta_{\bar{k},7_1} = -\delta_{\bar{k},7_2} = -\delta_{\bar{k},7_3} \equiv \delta_{\bar{k}} \,. \tag{B.20}$$

If the above conditions are satisfied, the tadpole conditions can be written in the form that appears in the main text, eqs. (4.7)–(4.12).

C. Shift formalism and discrete torsion

Let us sketch a general procedure for obtaining the open string spectrum of an arbitrary type-IIB orbifold or orientifold with discrete torsion. The method is based on the relation between the shift formalism and the one using matrices.

The open string spectrum of a general type-IIB orbifold with or without discrete torsion is determined by the matrices λ that satisfy

$$\gamma_g \lambda \gamma_g^{-1} = r(g)\lambda \,, \tag{C.1}$$

[15] This condition is necessary if one uses the standard Ω-projection of [36]. Using, however, the alternative projection of [37], this condition can be relaxed.

where γ_g represents the action of the orbifold group Γ on the Chan-Paton indices of the open string and $r(g)$ the action of Γ on the oscillator state of the open string. For the gauge degrees of freedom, $\lambda^{(0)}$, one has $r(g) = 1$, whereas $r(g) = e^{2\pi i v_i}$ for the matter degrees of freedom associated to the i-th complex plane, $\lambda^{(i)}$, if g acts as $z_i \to e^{2\pi i v_i} z_i$ on the i-th complex coordinate. A straightforward but cumbersome method to get the the spectrum, consists in constructing explicitely the matrices $\lambda^{(0)}$, $\lambda^{(i)}$ that solve (C.1). The $\lambda^{(0)}$ transform in the adjoint representation of the gauge group G. Knowing G, one finds the matter representations by looking at the transformation properties of the $\lambda^{(i)}$.

In [23] it has been shown that this is equivalent to a shift formalism which is very similar to the one that is known from heterotic orbifolds. If all the matrices γ_g commute, one can diagonalize them simultaneously by choosing the Cartan-Weyl basis:

$$\gamma_g = e^{-2\pi i V_g \cdot H}\,, \tag{C.2}$$

where $H = (H_1, H_2, \ldots, H_{\mathrm{rank}(\tilde{G})})$ is a vector containing the Cartan generators of the gauge group $\tilde{G}$ (before the orbifold projection) and $V_g = (V_1, V_2, \ldots, V_{\mathrm{rank}(\tilde{G})})$ is the shift vector that represents the action of $g \in \Gamma$ on the root lattice of $\tilde{G}$. One can show [23] that in the Cartan-Weyl basis the identity

$$\gamma_g E_a \gamma_g^{-1} = e^{-2\pi i \rho^a \cdot V_g} E_a\,, \qquad a = 1, \ldots, \dim(\tilde{G}) - \mathrm{rank}(\tilde{G})\,, \tag{C.3}$$

holds for all charged generators E_a of $\tilde{G}$, if ρ^a is the root vector associated to E_a, i.e. $[H_I, E_a] = \rho^a_I E_a$. The condition (C.1) now reads

$$\rho^a \cdot V_g = 0 \bmod \mathbb{Z} \quad \text{for gauge fields}\,, \tag{C.4}$$

$$\rho^a \cdot V_g = v_i \bmod \mathbb{Z} \quad \text{for matter fields from the } i\text{-th complex plane}\,. \tag{C.5}$$

In this representation, it is very easy to find the gauge group $G \subset \tilde{G}$ that is preserved after the orbifold projection and the representations of the matter fields.

If $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$ and the projective representation γ has discrete torsion, then $\gamma_{g_1}\gamma_{g_2} \neq \gamma_{g_2}\gamma_{g_1}$, where g_1, g_2 are the generators of $\mathbb{Z}_N, \mathbb{Z}_M$. The above method must be modified because γ_{g_1} and γ_{g_2} cannot be simultaneously diagonalized. Following the idea of [38, 39], we diagonalize γ_{g_1} and represent g_2 by a permutation acting on the entries of the root vectors.

For a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold with discrete torsion ϵ, the matrices γ_{g_1}, γ_{g_2} are of the following form (see appendix A):

$$\gamma_{g_1} = \bigoplus_k \left(\omega_N^k \gamma_{g_1}^{\mathrm{irrep}}\right) \otimes \mathbb{1}_{n_k}\,, \qquad \gamma_{g_2} = \bigoplus_l \left(\omega_M^l \gamma_{g_2}^{\mathrm{irrep}}\right) \otimes \mathbb{1}_{n_l}\,, \tag{C.6}$$

$$\text{with} \quad \gamma_{g_1}^{\mathrm{irrep}} = \mathrm{diag}\left(1, \epsilon^{-1}, \epsilon^{-2}, \ldots, \epsilon^{-(s-1)}\right), \qquad \gamma_{g_2}^{\mathrm{irrep}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & 0 & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix},$$

$$\omega_p = e^{2\pi i/p}\,, \qquad k = 0,\ldots,\frac{N}{s}-1\,, \qquad l = 0,\ldots,\frac{M}{s}-1\,,$$
s is the minimal positive integer, such that $\epsilon^s = 1$.

In the Cartan-Weyl basis, γ_{g_1} corresponds to the shift

$$\begin{aligned} V_{g_1} = \frac{1}{N}\Bigg(&0^{n_0}, \left(\frac{N}{s}\right)^{n_0}, \ldots, \left((s-1)\frac{N}{s}\right)^{n_0}, 1^{n_1}, \left(\frac{N}{s}+1\right)^{n_1}, \ldots, \left((s-1)\frac{N}{s}+1\right)^{n_1}, \\ &\ldots, \left(\frac{N}{s}-1\right)^{n_{N/s-1}}, \left(2\frac{N}{s}-1\right)^{n_{N/s-1}}, \ldots, (N-1)^{n_{N/s-1}}\Bigg). \end{aligned} \tag{C.7}$$

The matrix γ_{g_2} acts as a permutation Π_{g_2} on the root vectors ρ. We assume $N \geq M$ and consider first the case $s = M$, i.e. $\gamma_{g_2} = \gamma_{g_2}^{\rm irrep} \otimes \mathbb{1}_n$, where $n = \sum_k n_k$:

$$\begin{aligned} \Pi_{g_2}:\ &\left(\tilde\rho_1^{(n_0)},\ldots,\tilde\rho_s^{(n_0)},\tilde\rho_1^{(n_1)},\ldots,\tilde\rho_s^{(n_1)},\ldots,\tilde\rho_1^{(n_{N/s-1})},\ldots,\tilde\rho_s^{(n_{N/s-1})}\right) \\ &\longrightarrow \left(\tilde\rho_s^{(n_0)},\tilde\rho_1^{(n_0)},\ldots,\tilde\rho_{s-1}^{(n_0)},\tilde\rho_s^{(n_1)},\tilde\rho_1^{(n_1)},\ldots,\tilde\rho_{s-1}^{(n_1)},\ldots,\tilde\rho_s^{(n_{N/s-1})},\ldots,\tilde\rho_{s-1}^{(n_{N/s-1})}\right), \\ \text{where}\ \ \tilde\rho_i^{(n_k)} &= \left(\rho_{s(n_0+\cdots+n_{k-1})+(i-1)n_k+1},\ldots,\rho_{s(n_0+\cdots+n_{k-1})+in_k}\right). \end{aligned} \tag{C.8}$$

The gauge group G of the orbifold model is determined by finding all the root vectors ρ^a of $\mathrm{U}(sn)$ (or linear combinations of these) that satisfy

$$\rho^a \cdot V_{g_1} = 0 \bmod \mathbb{Z}\,, \qquad \Pi_{g_2}(\rho^a) = \rho^a\,. \tag{C.9}$$

If we choose twist vectors $v = \frac{1}{N}(1,-1,0)$ resp. $w = \frac{1}{M}(0,1,-1)$ for the action of g_1 resp. g_2 on $\mathbb{C}^3$, then the matter representations are obtained from the root vectors ρ^a of $\mathrm{U}(sn)$ (or linear combinations of these) that satisfy

$$\begin{aligned} \rho^a \cdot V_{g_1} &= \frac{1}{N} \bmod \mathbb{Z}\,, && \Pi_{g_2}(\rho^a) = \rho^a \quad \text{(1st complex plane)} \quad \text{or} \\ \rho^a \cdot V_{g_1} &= -\frac{1}{N} \bmod \mathbb{Z}\,, && \Pi_{g_2}(\rho^a) = e^{2\pi i/M}\rho^a \quad \text{(2nd complex plane)} \quad \text{or} \\ \rho^a \cdot V_{g_1} &= 0 \bmod \mathbb{Z}\,, && \Pi_{g_2}(\rho^a) = e^{-2\pi i/M}\rho^a \quad \text{(3rd complex plane)}. \end{aligned} \tag{C.10}$$

Consider, for example, $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion $\epsilon = \omega_N$, i.e. $s = N$ (these models were discussed in [6]). In this case, one has

$$\begin{aligned} V_{g_1} &= \frac{1}{N}\left(0^{n_0}, 1^{n_0}, \ldots, (N-1)^{n_0}\right), \\ \Pi_{g_2} &: \left(\tilde\rho_1^{(n_0)},\ldots,\tilde\rho_N^{(n_0)}\right) \longrightarrow \left(\tilde\rho_N^{(n_0)},\tilde\rho_1^{(n_0)},\ldots,\tilde\rho_{N-1}^{(n_0)}\right). \end{aligned} \tag{C.11}$$

The roots of $\mathrm{U}(Nn_0)$ are of the form $\rho^a = (\underline{+,-,0^{Nn_0-2}})$, where underlining means that all permutations have to be considered. It is easy to see that there are $n_0(n_0-1)$ linear combinations of roots ρ_a that satisfy (C.9):

$$\left(\underline{+,-,0^{n_0-2}},0^{(N-1)n_0}\right) + \left(0^{n_0},\underline{+,-,0^{n_0-2}},0^{(N-2)n_0}\right) + \cdots + \left(0^{(N-1)n_0},\underline{+,-,0^{n_0-2}}\right).$$

Together with the n_0 Cartan generators they form the gauge group $G = \mathrm{U}(n_0)$. Similarly, one finds $3\,n_0^2 - n_0$ linear combinations of roots ρ_a (and n_0 Cartan generators) that satisfy (C.10):

$$\begin{aligned}
\text{1st plane}: \quad & \big(\underline{-,0^{n_0-1}},\underline{+,0^{n_0-1}},0^{(N-2)n_0}\big) + \big(0^{n_0},\underline{-,0^{n_0-1}},\underline{+,0^{n_0-1}},0^{(N-3)n_0}\big) + \\
& + \cdots + \big(\underline{+,0^{n_0-1}},0^{(N-2)n_0},\underline{-,0^{n_0-1}}\big)\,, \\
\text{2nd plane}: \quad & \big(\underline{+,0^{n_0-1}},\underline{-,0^{n_0-1}},0^{(N-2)n_0}\big) + \alpha^{-1}\big(0^{n_0},\underline{+,0^{n_0-1}},\underline{-,0^{n_0-1}},0^{(N-3)n_0}\big) + \\
& + \cdots + \alpha^{-(N-1)}\big(\underline{-,0^{n_0-1}},0^{(N-2)n_0},\underline{+,0^{n_0-1}}\big)\,, \\
\text{3rd plane}: \quad & \big(\underline{+,-,0^{n_0-2}},0^{(N-1)n_0}\big) + \alpha\big(0^{n_0},\underline{+,-,0^{n_0-2}},0^{(N-2)n_0}\big) + \\
& + \cdots + \alpha^{N-1}\big(0^{(N-1)n_0},\underline{+,-,0^{n_0-2}}\big)\,,
\end{aligned}$$

where $\alpha = e^{2\pi i/N}$. These are three matter fields in the adjoint representation of $\mathrm{U}(n_0)$.

As a second example, consider $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_2$ with discrete torsion $\epsilon = -1$, i.e. $s = 2$. In this case, one has

$$\begin{aligned}
V_{g_1} &= \frac{1}{N}\left(0^{n_0},\left(\frac{N}{2}\right)^{n_0},1^{n_1},\left(\frac{N}{2}+1\right)^{n_1},\ldots,\left(\frac{N}{2}-1\right)^{n_{N/2-1}},(N-1)^{n_{N/2-1}}\right), \\
\Pi_{g_2} &: \left(\tilde{\rho}_1^{(n_0)},\tilde{\rho}_2^{(n_0)},\tilde{\rho}_1^{(n_1)},\tilde{\rho}_2^{(n_1)},\ldots,\tilde{\rho}_1^{(n_{N/2-1})},\tilde{\rho}_2^{(n_{N/2-1})}\right) \\
&\quad \longrightarrow \left(\tilde{\rho}_2^{(n_0)},\tilde{\rho}_1^{(n_0)},\tilde{\rho}_2^{(n_1)},\tilde{\rho}_1^{(n_1)},\ldots,\tilde{\rho}_2^{(n_{N/2-1})},\tilde{\rho}_1^{(n_{N/2-1})}\right). \qquad \text{(C.12)}
\end{aligned}$$

There are $n_0(n_0-1) + n_1(n_1-1) + \cdots + n_{N/2-1}(n_{N/2-1}-1)$ linear combinations of roots of $\mathrm{U}(2n)$ that satisfy (C.9). Together with the $n = \sum_k n_k$ Cartan generators, they form the gauge group $G = \mathrm{U}(n_0) \times \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{N/2-1})$. Similarly, one finds the following matter representations from the 3 complex planes:

$$\begin{aligned}
\text{1st plane}: &\ (\overline{\Box},\Box,\mathbb{1},\ldots,\mathbb{1}) + (\mathbb{1},\overline{\Box},\Box,\mathbb{1},\ldots,\mathbb{1}) + \cdots + (\mathbb{1},\ldots,\mathbb{1},\overline{\Box},\Box) + (\Box,\mathbb{1},\ldots,\mathbb{1},\overline{\Box}) \\
\text{2nd plane}: &\ (\Box,\overline{\Box},\mathbb{1},\ldots,\mathbb{1}) + (\mathbb{1},\Box,\overline{\Box},\mathbb{1},\ldots,\mathbb{1}) + \cdots + (\mathbb{1},\ldots,\mathbb{1},\Box,\overline{\Box}) + (\overline{\Box},\mathbb{1},\ldots,\mathbb{1},\Box) \\
\text{3rd plane}: &\ (\mathrm{adj},\mathbb{1},\ldots,\mathbb{1}) + (\mathbb{1},\mathrm{adj},\mathbb{1},\ldots,\mathbb{1}) + \cdots + (\mathbb{1},\ldots,\mathbb{1},\mathrm{adj})\,.
\end{aligned}$$

It is instructive to see in a simple special case that the permutation Π_{g_2} in the shift formalism indeed corresponds to the action of γ_{g_2} in the formalism using matrices. Let us take $\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2$. According to (C.6) the matrices γ_{g_1}, γ_{g_2} are given by

$$\gamma_{g_1} = \begin{pmatrix} \mathbb{1}_{n_0} & 0 & 0 & 0 \\ 0 & -\mathbb{1}_{n_0} & 0 & 0 \\ 0 & 0 & i\mathbb{1}_{n_1} & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_{n_1} \end{pmatrix}, \qquad \gamma_{g_2} = \begin{pmatrix} 0 & \mathbb{1}_{n_0} & 0 & 0 \\ \mathbb{1}_{n_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{n_1} \\ 0 & 0 & \mathbb{1}_{n_1} & 0 \end{pmatrix}. \qquad \text{(C.13)}$$

As γ_{g_1} is diagonal, according to (C.2), it corresponds to a shift $V_{g_1} = \frac{1}{4}(0^{n_0}, 2^{n_0}, 1^{n_1}, 3^{n_1})$. The action of $\gamma_{g_2} E_a \gamma_{g_2}^{-1}$ on the matrix E_a is such that it permutes the associated roots ρ^a as $(\tilde\rho_1^{(n_0)},\tilde\rho_2^{(n_0)},\tilde\rho_1^{(n_1)},\tilde\rho_2^{(n_1)}) \to (\tilde\rho_2^{(n_0)},\tilde\rho_1^{(n_0)},\tilde\rho_2^{(n_1)},\tilde\rho_1^{(n_1)})$. This coincides with the prescription given above. More precisely, $\gamma_{g_2}E_a\gamma_{g_2}^{-1} = \alpha E'_a$, where E'_a is the matrix corresponding to the permuted root vector. The phase $\alpha = e^{-m\pi i/M}$, $m = 0,1,2,3$, is related to an ambiguity in choosing a specific basis for the matrices E_a. It does not affect the orbifold spectrum and can therefore be ignored. However, in the orientifold case, it will be important to take this phase into account.

For a general $\mathbb{Z}_N\times\mathbb{Z}_M$ orbifold with discrete torsion ϵ, such that $s<M$, one has to take M/s copies of the shift vector (C.7), with entries labelled by

$$\Big(n_{0,0},n_{1,0},\ldots,n_{N/s-1,0}|n_{0,1},n_{1,1},\ldots,n_{N/s-1,1}|\cdots|n_{0,M/s-1},\\ n_{1,M/s-1},\ldots,n_{N/s-1,M/s-1}\Big)\,.$$

The permutation Π_{g_2} acts on each copy identically, as in (C.8). To represent the action of the phases ω_M^l that appear in (C.6) for $s<M$, one has to associate a shift vector also to the element g_2:

$$V_{g_2} = \frac{1}{M}\left(0^{\tilde n_0},1^{\tilde n_1},\ldots,\left(\frac{M}{s}-1\right)^{\tilde n_{M/s-1}}\right)\,,\quad \text{with } \tilde n_l = \sum_k n_{k,l}\,. \tag{C.14}$$

To determine the gauge group, one has to find all roots ρ^a of $\mathrm{U}(s\sum_{k,l}n_{k,l})$ that satisfy

$$\rho^a\cdot V_{g_1} = 0 \bmod \mathbb{Z}\,,\qquad \rho^a\cdot V_{g_2} = 0 \bmod \mathbb{Z}\,,\qquad \Pi_{g_2}(\rho^a) = \rho^a\,. \tag{C.15}$$

The roots corresponding to the matter representations have to satisfy, for each complex plane, *one* of the following s conditions:

$$\begin{aligned} &\rho^a\cdot V_{g_1} = v_i \bmod \mathbb{Z}\,,\qquad \rho^a\cdot V_{g_2} = w_i - \frac{r}{s} \bmod \mathbb{Z}\,,\qquad \Pi_{g_2}(\rho^a) = e^{2\pi i r/s}\rho^a\,,\\ &i=1,2,3\,,\ r=0,\ldots,s-1\,,\\ &v_1 = \frac{1}{N}\,,\ v_2 = -\frac{1}{N}\,,\ v_3 = 0\,,\ w_1 = 0\,,\ w_2 = \frac{1}{M}\,,\ w_3 = -\frac{1}{M}\,. \end{aligned} \tag{C.16}$$

As an example, let us compute the spectrum of the $\mathbb{Z}_6\times\mathbb{Z}_6$ orbifold with discrete torsion $\epsilon = e^{2\pi i/3}$, i.e. $s=3$. The shift vectors and permutation are given by

$$\begin{aligned} V_{g_1} &= \frac{1}{6}\Big(0^{n_{0,0}},2^{n_{0,0}},4^{n_{0,0}},1^{n_{1,0}},3^{n_{1,0}},5^{n_{1,0}}|0^{n_{0,1}},2^{n_{0,1}},4^{n_{0,1}},1^{n_{1,1}},3^{n_{1,1}},5^{n_{1,1}}\Big)\,,\\ V_{g_2} &= \frac{1}{6}\Big(0^{\tilde n_0},1^{\tilde n_1}\Big)\,,\\ \Pi_{g_2}\ &:\ (\tilde\rho_1^{n_{0,0}},\tilde\rho_2^{n_{0,0}},\tilde\rho_3^{n_{0,0}},\ldots,\tilde\rho_1^{n_{1,1}},\tilde\rho_2^{n_{1,1}},\tilde\rho_3^{n_{1,1}})\\ &\qquad\longrightarrow(\tilde\rho_3^{n_{0,0}},\tilde\rho_1^{n_{0,0}},\tilde\rho_2^{n_{0,0}},\ldots,\tilde\rho_3^{n_{1,1}},\tilde\rho_1^{n_{1,1}},\tilde\rho_2^{n_{1,1}})\,. \end{aligned} \tag{C.17}$$

There are $n_{0,0}(n_{0,0}-1)$ linear combinations of roots,

$$\Big(\underline{+,-,0^{n_{0,0}-2}},0,\ldots,0\Big)+\Big(0^{n_{0,0}},\underline{+,-,0^{n_{0,0}-2}},0,\ldots,0\Big)+\Big(0^{2n_{0,0}},\underline{+,-,0^{n_{0,0}-2}},0,\ldots,0\Big)\,,$$

that satisfy (C.15), and similarly for $n_{1,0}$, $n_{0,1}$, $n_{1,1}$. Together with the Cartan generators, they form the gauge group $\mathrm{U}(n_{0,0}) \times \mathrm{U}(n_{1,0}) \times \mathrm{U}(n_{0,1}) \times \mathrm{U}(n_{1,1})$. Concerning the matter fields from the first plane, only $r = 0$ is possible in (C.16). This gives the following representations:

$$(\Box, \overline{\Box}, \mathbb{1}, \mathbb{1}) + (\overline{\Box}, \Box, \mathbb{1}, \mathbb{1}) + (\mathbb{1}, \mathbb{1}, \Box, \overline{\Box}) + (\mathbb{1}, \mathbb{1}, \overline{\Box}, \Box)\,.$$

In the second plane, $r = 0$ and $r = 1$ are possible, yielding

$$r = 0: \ (\mathbb{1}, \overline{\Box}, \Box, \mathbb{1}) + (\overline{\Box}, \mathbb{1}, \mathbb{1}, \Box)\,, \qquad r = 1: \ (\mathbb{1}, \Box, \overline{\Box}, \mathbb{1}) + (\Box, \mathbb{1}, \mathbb{1}, \overline{\Box})\,.$$

In the third plane, $r = 0$ and $r = 2$ are possible, yielding

$$r = 0: \ (\Box, \mathbb{1}, \overline{\Box}, \mathbb{1}) + (\mathbb{1}, \Box, \mathbb{1}, \overline{\Box})\,, \qquad r = 2: \ (\overline{\Box}, \mathbb{1}, \Box, \mathbb{1}) + (\mathbb{1}, \overline{\Box}, \mathbb{1}, \Box)\,.$$

The open string spectrum of a general type-IIB orientifold with or without discrete torsion is determined by the matrices λ that satisfy (C.1) and, in addition,

$$\gamma_\Omega \lambda^\top \gamma_\Omega^{-1} = r(\Omega)\lambda\,, \tag{C.18}$$

where γ_Ω represents the action of the world-sheet parity Ω on the Chan-Paton indices of the open string and $r(\Omega)$ the action of Ω on the oscillator state of the open string. In complete analogy to the orbifold case, the spectrum can equivalently be obtained using the shift formalism. The additional condition (C.18) implies that the gauge group $\tilde{G}$ before the orbifold projection is $\mathrm{SO}(4\sum_{k,l} n_{k,l})$ resp. $\mathrm{USp}(4\sum_{k,l} n_{k,l})$ if γ_Ω is symmetric resp. antisymmetric. In the case of orientifolds with discrete torsion, one has to find all root vectors ρ^a of $\tilde{G}$ that satisfy (C.15) and (C.16). As a consequence of $\Omega g \Omega = g$, the matrices γ_g now form a real projective representation and only discrete torsion $\epsilon = \pm 1$ is possible.

In this case one must also distinguish between orientifolds with four different boundary conditions on the gauge bundle: $\gamma_{g_1}^N = \pm\mathbb{1}$, $\gamma_{g_2}^M = \pm\mathbb{1}$. In the shift formalism this translates to

$$NV_{g_1} = \begin{cases} (1, \ldots, 1) \bmod \mathbb{Z} & \text{if } \gamma_{g_1}^N = \mathbb{1} \\ \frac{1}{2}(1, \ldots, 1) \bmod \mathbb{Z} & \text{if } \gamma_{g_1}^N = -\mathbb{1}\,, \end{cases} \tag{C.19}$$

and similarly for MV_{g_2}. As mentioned above (below eq. (C.13)), there is an ambiguity in the choice of a basis for the Chan-Paton matrices λ which leads to an additional phase in the action of g_2 on λ. It turns out that the above prescription, using a shift V_{g_2} of the form (C.14) and requiring the conditions (C.15), (C.16), corresponds in the orientifold case to $\gamma_{g_2}^M = -\mathbb{1}$. However, using the shift V_{g_1} of the form (C.7) for the group element g_1, corresponds to $\gamma_{g_1}^N = \mathbb{1}$, as expected. If $\gamma_{g_1}^N = -\mathbb{1}$, $\gamma_{g_2}^M = \mathbb{1}$, then the general form of the shifts given above, eqs. (C.7), (C.14), is modified to (using $s = 2$ and taking $M/2$ copies of (C.7))

$$\begin{aligned}
V_{g_1} &= \frac{1}{2N}\Big(1^{n_{0,0}}, (N+1)^{n_{0,0}}, 3^{n_{1,0}}, (N+3)^{n_{1,0}}, \ldots, (N-1)^{n_{N/2-1,0}}, (2N-1)^{n_{N/2-1,0}} | \\
&\qquad 1^{n_{0,1}}, (N+1)^{n_{0,1}}, \ldots | \cdots | \ldots (2N-1)^{n_{N/2-1,M/2-1}}\Big)\,, \\
V_{g_2} &= \frac{1}{2M}\Big(1^{\tilde{n}_0}, 3^{\tilde{n}_1}, \ldots, (M-1)^{\tilde{n}_{M/2-1}}\Big)\,, \quad \text{with } \tilde{n}_l = \sum_k n_{k,l}\,.
\end{aligned} \tag{C.20}$$

The permutation Π_{g_2} is not modified. The gauge group is determined by the roots ρ^a that satisfy *one* of the two conditions

$$\begin{aligned}
&\rho^a \cdot V_{g_1} = 0 \bmod \mathbb{Z}\,, \qquad \rho^a \cdot V_{g_2} = \frac{r}{2} \bmod \mathbb{Z}\,, \\
&\Pi_{g_2}(\rho^a) = (-1)^r \rho^a\,, \quad r = 0, 1\,.
\end{aligned} \tag{C.21}$$

The matter fields are obtained from (C.16) by setting $s = 2$.

References

[1] C. Vafa, *Modular invariance and discrete torsion on orbifolds*, *Nucl. Phys.* **B 273** (1986) 592.

[2] A. Font, L.E. Ibáñez and F. Quevedo, $\mathbb{Z}_N \times \mathbb{Z}_M$ *orbifolds and discrete torsion*, *Phys. Lett.* **B 217** (1989) 272.

[3] C. Vafa and E. Witten, *On orbifolds with discrete torsion*, *J. Geom. Phys.* **15** (1995) 189 [hep-th/9409188].

[4] P.S. Aspinwall, D.R. Morrison and M. Gross, *Stable singularities in string theory*, *Comm. Math. Phys.* **178** (1996) 115 [hep-th/9503208].

[5] M.R. Douglas, *D-branes and discrete torsion*, hep-th/9807235.

[6] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, 2*, hep-th/9903031.

[7] J. Gomis, *D-branes on orbifolds with discrete torsion and topological obstruction*, *J. High Energy Phys.* **05** (2000) 006 [hep-th/0001200].

[8] D. Berenstein and R.G. Leigh, *Discrete torsion, AdS/CFT and duality*, *J. High Energy Phys.* **01** (2000) 038 [hep-th/0001055].

[9] Z. Kakushadze, G. Shiu and S.H.H. Tye, *Type-IIB orientifolds with NS-NS antisymmetric tensor backgrounds*, *Phys. Rev.* **D 58** (1998) 086001 [hep-th/9803141].

[10] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365;
M. Bianchi, *A note on toroidal compactifications of the type-I superstring and other superstring vacuum configurations with 16 supercharges*, *Nucl. Phys.* **B 528** (1998) 73 [hep-th/9711201].

[11] C. Angelantonj, *Comments on open-string orbifolds with a non-vanishing B_{ab}*, *Nucl. Phys.* **B 566** (2000) 126 [hep-th/9908064];
C. Angelantonj and R. Blumenhagen, *Discrete deformations in type-I vacua*, *Phys. Lett.* **B 473** (2000) 86 [hep-th/9911190].

[12] Z. Kakushadze, *Geometry of orientifolds with NS-NS B-flux*, hep-th/0001212.

[13] E. Witten, *Toroidal compactification without vector structure*, *J. High Energy Phys.* **02** (1998) 006 [hep-th/9712028].

[14] M. Cvetič, M. Plumacher and J. Wang, *Three family type-IIB orientifold string vacua with non-abelian Wilson lines*, *J. High Energy Phys.* **04** (2000) 004 [hep-th/9911021].

[15] E.R. Sharpe, *Discrete torsion and gerbes, 1*, hep-th/9909108.

[16] C. Angelantonj, I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, *Type-I vacua with brane supersymmetry breaking*, *Nucl. Phys.* **B 572** (2000) 36 [hep-th/9911081].

[17] M. Klein, R. Rabadán, *$\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds with and without discrete torsion*, hep-th/0008173.

[18] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[19] G. Karpilovsky, *Projective representations of finite groups*, M. Dekker, New York 1985.

[20] M. Hamermesh, *Group theory and its application to physical problem*, Addison-Wesley, Reading 1964.

[21] M.R. Douglas and G. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[22] J. Louis and K. Förger, *Holomorphic couplings in string theory*, *Nucl. Phys.* **55B** *(Proc. Suppl.)* (1997) 33 [hep-th/9611184].

[23] G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, *$D = 4$, $N = 1$, type-IIB orientifolds*, *Nucl. Phys.* **B 536** (1998) 29 [hep-th/9804026].

[24] R.G. Leigh and M. Rozali, *Brane boxes, anomalies, bending and tadpoles*, *Phys. Rev.* **D 59** (1999) 026004 [hep-th/9807082].

[25] G. Aldazabal, D. Badagnani, L.E. Ibáñez and A.M. Uranga, *Tadpole versus anomaly cancellation in $D = 4$, 6 compact IIB orientifolds*, *J. High Energy Phys.* **06** (1999) 031 [hep-th/9904071].

[26] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[27] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[28] D. Joyce, *On the topology of desingularizations of Calabi-Yau orbifolds*, math.AG/9806146.

[29] S. Mukhopadhyay and K. Ray, *D-branes on fourfolds with discrete torsion*, *Nucl. Phys.* **B 576** (2000) 152 [hep-th/9909107].

[30] J. Park, R. Rabadán and A.M. Uranga, *Orientifolding the conifold*, *Nucl. Phys.* **B 570** (2000) 38 [hep-th/9907086].

[31] M. Klein, *Anomaly cancellation in $D = 4$, $N = 1$ orientifolds and linear/chiral multiplet duality*, *Nucl. Phys.* **B 569** (2000) 362 [hep-th/9910143].

[32] G. Zwart, *Four-dimensional $N = 1$ $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds*, *Nucl. Phys.* **B 526** (1998) 378 [hep-th/9708040].

[33] Z. Kakushadze and G. Shiu, *4d chiral $N = 1$ type-I vacua with and without D5-branes*, *Nucl. Phys.* **B 520** (1998) 75 [hep-th/9706051].

[34] M. Berkooz, R.G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and E. Witten, *Anomalies, dualities and topology of $D = 6$ $N = 1$ superstring vacua*, *Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184].

[35] M. Berkooz and R.G. Leigh, *A $D = 4$ $N = 1$ orbifold of type-I strings*, *Nucl. Phys.* **B 483** (1997) 187 [hep-th/9605049].

[36] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[37] A. Dabholkar and J. Park, *A note on orientifolds and F-theory*, *Phys. Lett.* **B 394** (1997) 302 [hep-th/9607041].

[38] G. Aldazabal, A. Font, L.E. Ibáñez and A.M. Uranga, *String GUTs*, *Nucl. Phys.* **B 452** (1995) 3 [hep-th/9410206].

[39] G. Aldazabal, A. Font, L.E. Ibáñez, A.M. Uranga and G. Violero, *Non-perturbative heterotic $D = 6, 4$, $N = 1$ orbifold vacua*, *Nucl. Phys.* **B 519** (1998) 239 [hep-th/9706158].

[40] A. Font, L.E. Ibáñez and F. Quevedo, *Higher level Kac-Moody string models and their phenomenological implications*, *Nucl. Phys.* **B 345** (1990) 389.